gravity.

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GR

Lecture 19

① $$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 - \frac{2GM}{r} & & & 0 \\ & -\left(1 - \frac{2GM}{r}\right)^{-1} & & \\ & & -r^2 & \\ 0 & & & -r^2 \sin^2\theta \end{pmatrix}$$

② Central forces in Newtonian

$\vec{F} = f(r)\hat{r}$ (function only depends on r & is in r direction only)

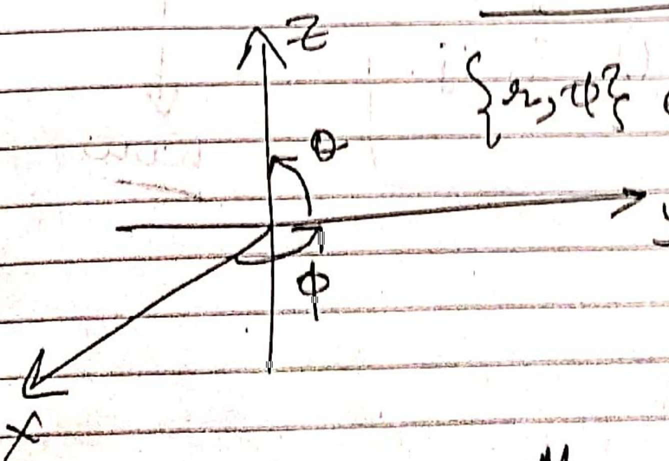
$\Downarrow$

$$\vec{\tau} = \vec{r} \times \vec{F} = 0 = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \text{const} = mr^2\omega$$

$\omega = \dot{\phi}$.

$\therefore$ Ang. Mom is $\perp^r$ to this plane $\Leftarrow$ Motion of Body should lie in plane

$\Downarrow$



$\{r, \phi\} \leftarrow$ $\therefore$ we can use 2 coordinates only Assuming $\theta = \text{const} = \pi/2$

$\therefore$ in XY plane

No generality lose.

③ System = source (M) + test mass (m)

E_{tot} = constant as system is closed

$$E_{tot} = \frac{m}{2}v^2 + V(r).$$

$x = r\cos\phi$
$y = r\sin\phi$

$\frac{d\hat{r}}{dt}$

(4) $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j}) = \frac{d}{dt}(r\hat{r})$

$$x = f(r, \theta) \Rightarrow \dot{x} = \frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial \theta}\dot{\theta}$$
$$y = g(r, \theta) \Rightarrow \dot{y} = \frac{\partial g}{\partial r}\dot{r} + \frac{\partial g}{\partial \theta}\dot{\theta}$$

$$\frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 \Rightarrow \frac{m}{2}\left[\left(\frac{\partial f}{\partial r}^2 + \frac{\partial g}{\partial r}^2\right)\dot{r}^2 + \left(\frac{\partial f}{\partial \theta}^2 + \frac{\partial g}{\partial \theta}^2\right)\dot{\theta}^2 + 2\left(\frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta} + \frac{\partial g}{\partial r}\frac{\partial g}{\partial \theta}\right)\dot{r}\dot{\theta}\right]$$

$$\Rightarrow \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + 2\left(\cancel{-r\cos\theta\sin\phi + r\sin\phi\cos\theta}\right)\right]$$

$$\Rightarrow \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2$$

$$\frac{m}{2}v^2 \Rightarrow \quad \vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j})$$

$$\frac{d}{dt}(r\hat{r})$$

now

$$\boxed{K.E. \Rightarrow \frac{m}{2}\sum_{ij} a_{ij}(q)\,\dot{q}_i\,\dot{q}_j}$$

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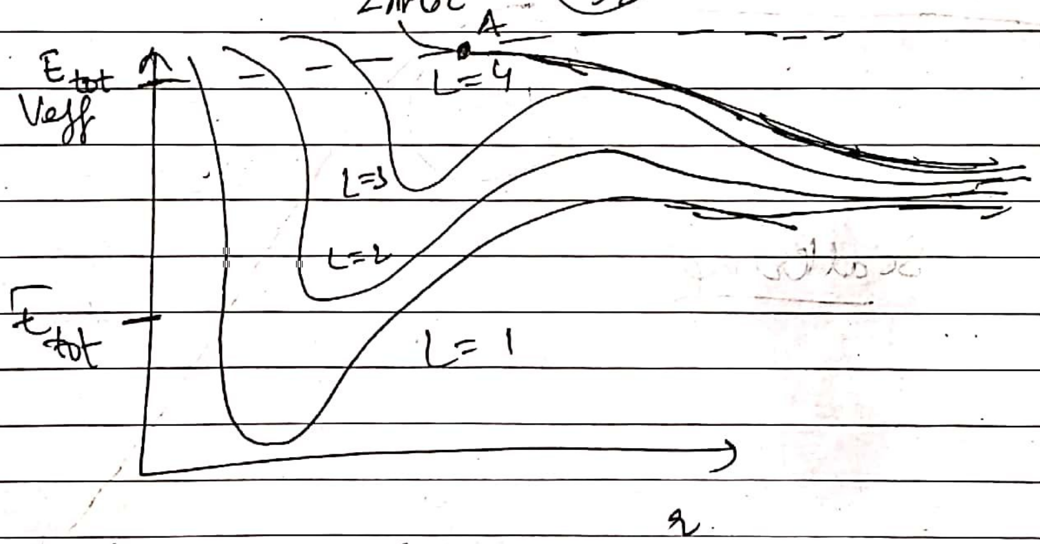
(5) $\therefore E_{tot} = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} + V(r) = \text{Const}$

$$= \frac{m\dot{r}^2}{2} + \underbrace{\frac{L^2}{2mr^2} + V(r)}_{V_{eff}(r)}$$

This is DE in (r).

(6) Let $V(r) = -\frac{GM}{r}$ → Source

$$\therefore V_{eff} = \frac{L^2}{2mr^2} - \frac{GM}{r}$$



As E_{tot} is Const. I can give it any amount & as L is also const. & ind. of E_{tot} it can take any value for any given E_{tot}.

We can also see $V_{eff} > 0 \;\forall\, r$

$$\therefore E_{tot} = \frac{mv^2}{2} + V_{eff} \quad +ve$$

at pt. A $E_{tot} = V_{eff}$ $\therefore$ K.E. = 0

~~But partical can go down V_{eff}.~~

Bec. KE > 0 $\therefore$ V_{eff} dec. & K.E ↑

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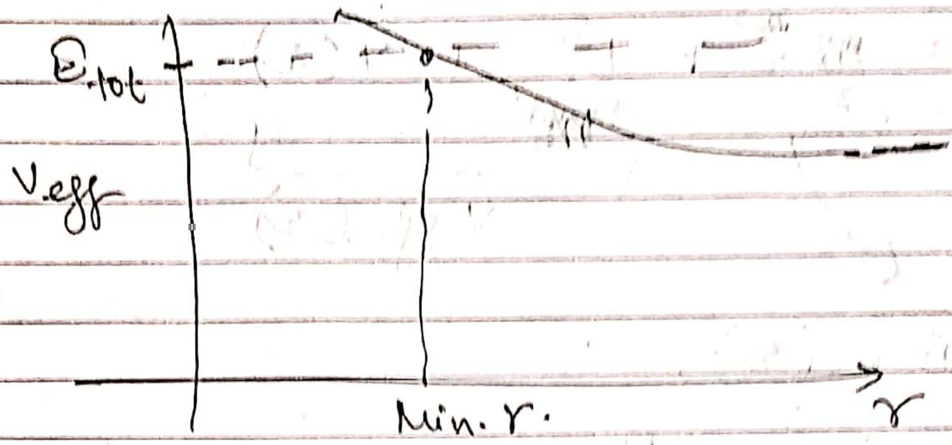
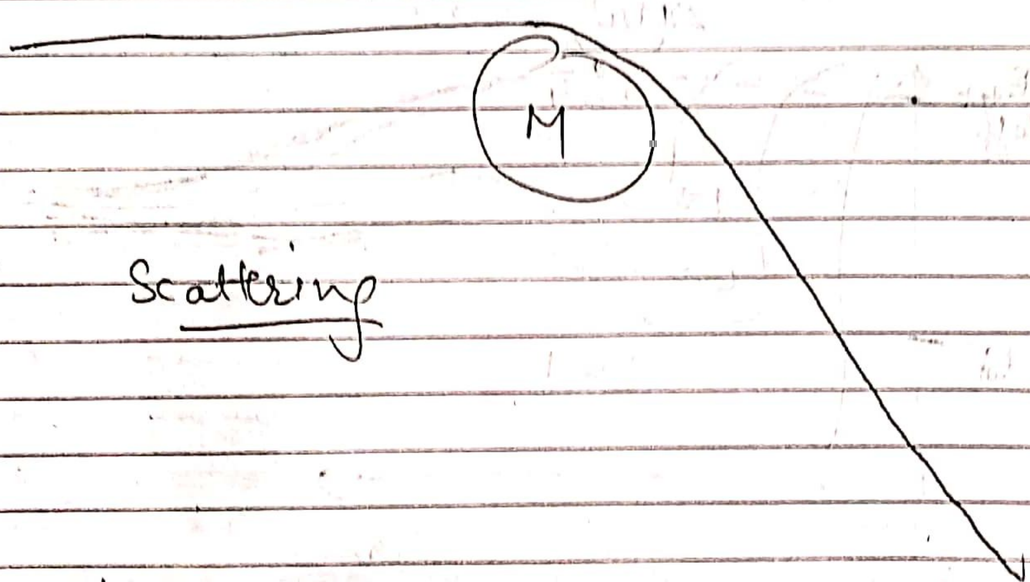
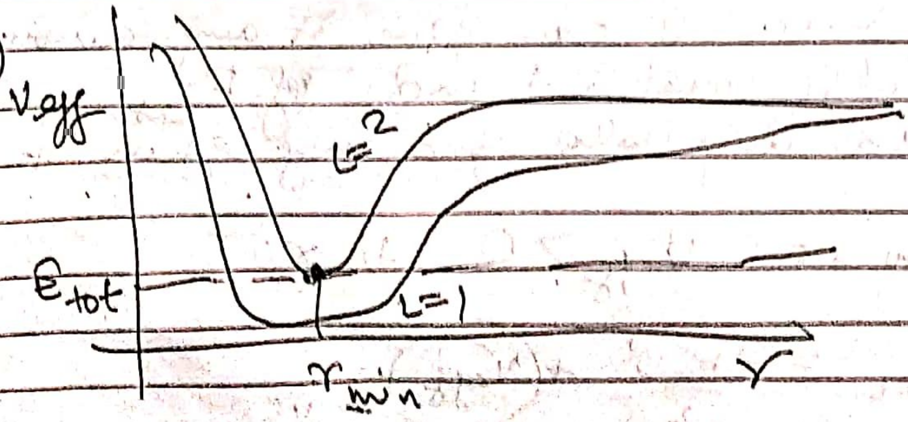
(7) In this case





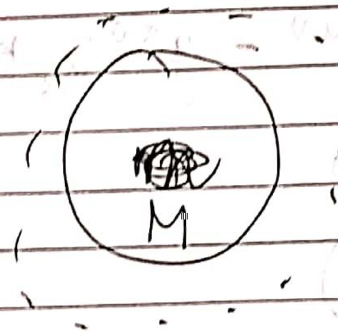
Scattering

(8)



In L=2 case (Stable)

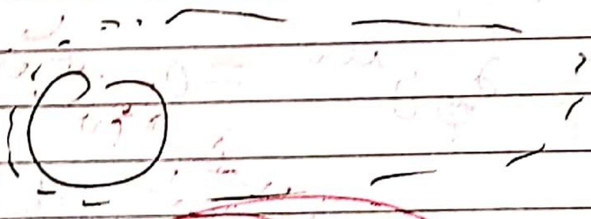
We are fixed at r_{min} ∴ Circular Orbit



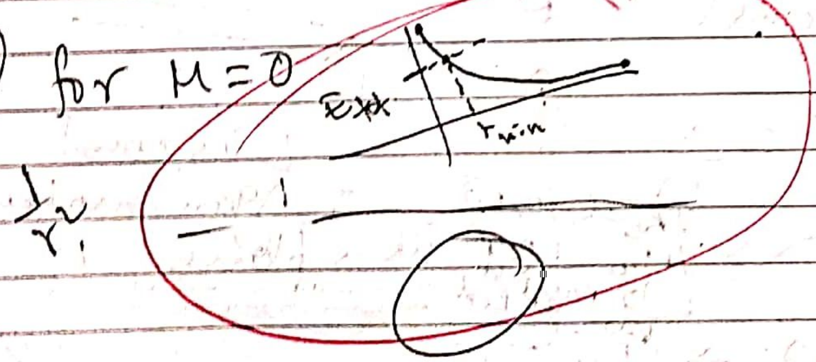
→ But $V=0$? Doubt

In L=1 case. (Stable)

Ellipse



(9) for $M=0$



Doubt

How to perdict Mathematically?

(10) $g^{\mu\nu} = (g_{\mu\nu})^{-1}$

(11) In Newtonian, constant is Energy (conser)
In GR, use killing vector ⇒ Const of Motion

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(12) $g_{\mu\nu}$ is Ind. of time, ϕ $\quad g_{\mu\nu}(r,\theta)$

$$\partial_t g_{\mu\nu} = 0$$

$$\mathcal{L}_K g^{\mu\nu} = K^\alpha \partial_\alpha g^{\mu\nu} - g^{\alpha\nu}\partial_\alpha K^\mu - g^{\mu\alpha}\partial_\alpha K^\nu$$

$$K_t = (1,0,0,0)$$

$$K_\phi = (0,0,0,1)$$

$$\therefore \mathcal{L}_K g^{\mu\nu} = K^\alpha \partial_\alpha g^{\mu\nu}$$

$$\mathcal{L}_{K_t} g^{\mu\nu} = \partial_0 g^{\mu\nu} = 0$$

$$\mathcal{L}_{K_\phi} g^{\mu\nu} = \partial_\phi g^{\mu\nu} = 0$$

What about Null Curves?

let $\theta = \pi/2$

(13) $K^\mu U_\mu = \text{const}$

$$K_t^\mu U_\mu = \tilde{E} = \frac{E}{m}$$

$$U^\mu = \frac{dx^\mu}{d\lambda}$$

$K_t = (1,0,0,0)$
$U_\mu = (1,0,0,0)$
$\hookrightarrow$ Particle at Rest

$$U_\mu = g_{\mu\alpha}\frac{dx^\alpha}{d\lambda} \Rightarrow \therefore \tilde{E} = g_{t\alpha}\frac{dx^\alpha}{d\lambda}$$

$$K_t^\mu U_\mu = -\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} = \tilde{E}$$

EOM for BH

Γ^t_{01}

$$K_\phi^\mu U_\mu = \tilde{L} = K_\phi^\mu g_{\mu\alpha}\frac{dx^\alpha}{d\lambda}$$

$$\tilde{L} = g_{\phi\phi}\frac{d\phi}{d\lambda} = r^2 \sin^2\theta \frac{d\phi}{d\lambda}$$

L is const

$\therefore$ Again motion in a plane

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$g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$

$dt^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$1 = g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$

(12) (14) $\epsilon = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$

$\frac{D\epsilon}{d\lambda} = 0 \Rightarrow \epsilon = \text{const}.$

(15) $\epsilon = \left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2$

Doing similar thing as in Newtonian.

$\left(\frac{dt}{d\lambda}\right)$ & $\left(\frac{d\phi}{d\lambda}\right)$ remove by const E & L

& get Eq. in dr terms.

(13) $\epsilon = \frac{E^2}{m^2\left(1-\frac{2GM}{r}\right)} - \frac{(dr/d\lambda)^2}{1-\frac{2GM}{r}} - \frac{L^2}{m r^2}$ D.E. in r

In Newtonian $E_{TOT} = \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff}(r)$

for $m>0$ $\epsilon = 1$

$m = 0$ $\epsilon = 0$

$\frac{E^2}{2m^2} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\epsilon}{2} - \frac{GM}{r}\epsilon + \frac{L^2}{2r^2 m} - \frac{GML^2}{r^3}$

$m = 0$ $\epsilon = 0$

$\frac{E^2}{2} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{L^2}{2r^2} - \frac{GML^2}{r^3}}_{V_{eff}}$

for general Veff

$\frac{dV_{eff}}{dr} = 0 \Rightarrow L^2(r-3M)$

GMr^2

(1) $\exists$ $r \geq 3M$

(2) $E=0$ $r = 3M$

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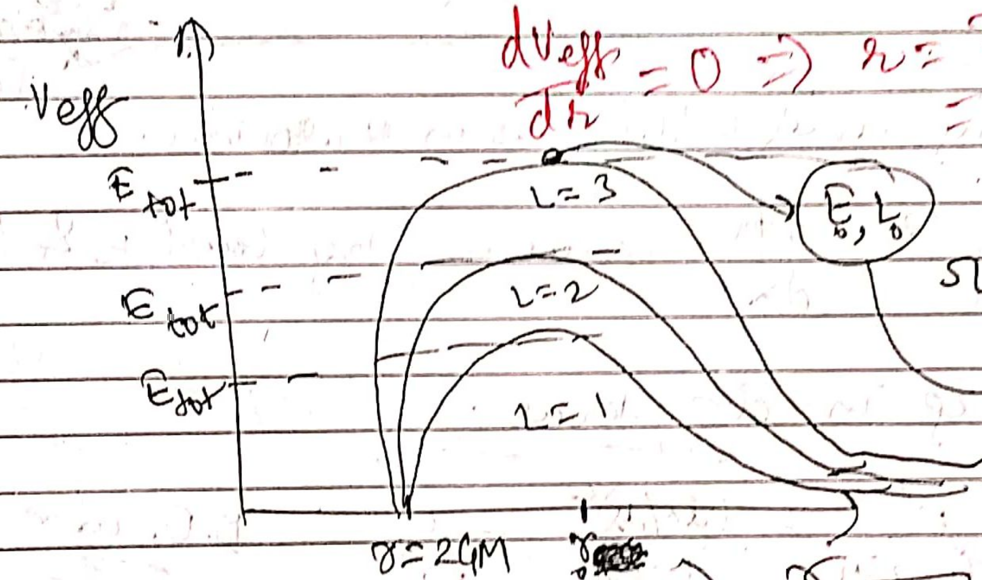
In Newton $V_{eff} = \frac{L^2}{2r^2} - \frac{GM}{r}$ for $M=0$

$$V_{eff} = \frac{L^2}{2r^2}$$

But in GR

$$V_{eff} = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$\frac{dV_{eff}}{dr} = 0 \Rightarrow r = 3GM = \frac{3}{2} r_s$



$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{fL}{r^2E}$

$\Omega^2 = \frac{M}{r_0^3}$

By putting value of L_0

Rel. Analog of Kepler's law

stable $\Rightarrow r > 6M$

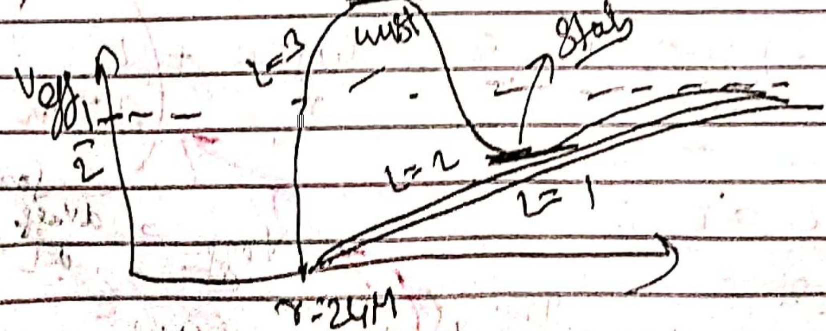
V_{eff}'' at r_0
$V_{eff}'' > 0$ stable
$V_{eff}'' < 0$ Unstable

(All Unstable)

$$V_{eff}'' = -\frac{2GM(r-6M)}{r^3(r-3M)}$$

∴ Massless particle going circular Orbit

Ex $M>0$

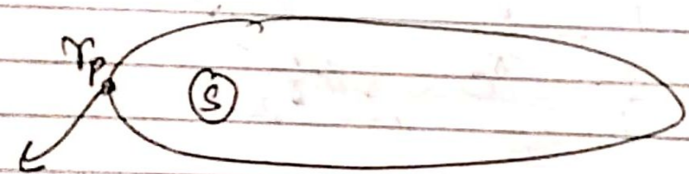


Critical Radius Radius

(20)

Earth	0.03 m	6×10^6 m
Sun	8850 m	7×10^8 m
White Dwarf	8850 m	10^6 m
Newton Star	8850	10^4 m
BH	8850	0

(21) Perihelion Shift



Perihelion

(22) Newtonian Case: $E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2}$

Using $L = r^2 \frac{d\phi}{dt} \Rightarrow dt = \frac{r^2 d\phi}{L}$

$$\left(\frac{dr}{d\phi}\right)^2 - \frac{2GM}{L^2} r^3 + r^2 = \frac{2E}{L^2} r^4 \Rightarrow r(\phi)$$

$$r(\phi) = \frac{L^2}{GM(1+e\cos\phi)}$$

$e(G, M, L) = \text{const}$

$r(\phi + 2\pi) = r(\phi)$

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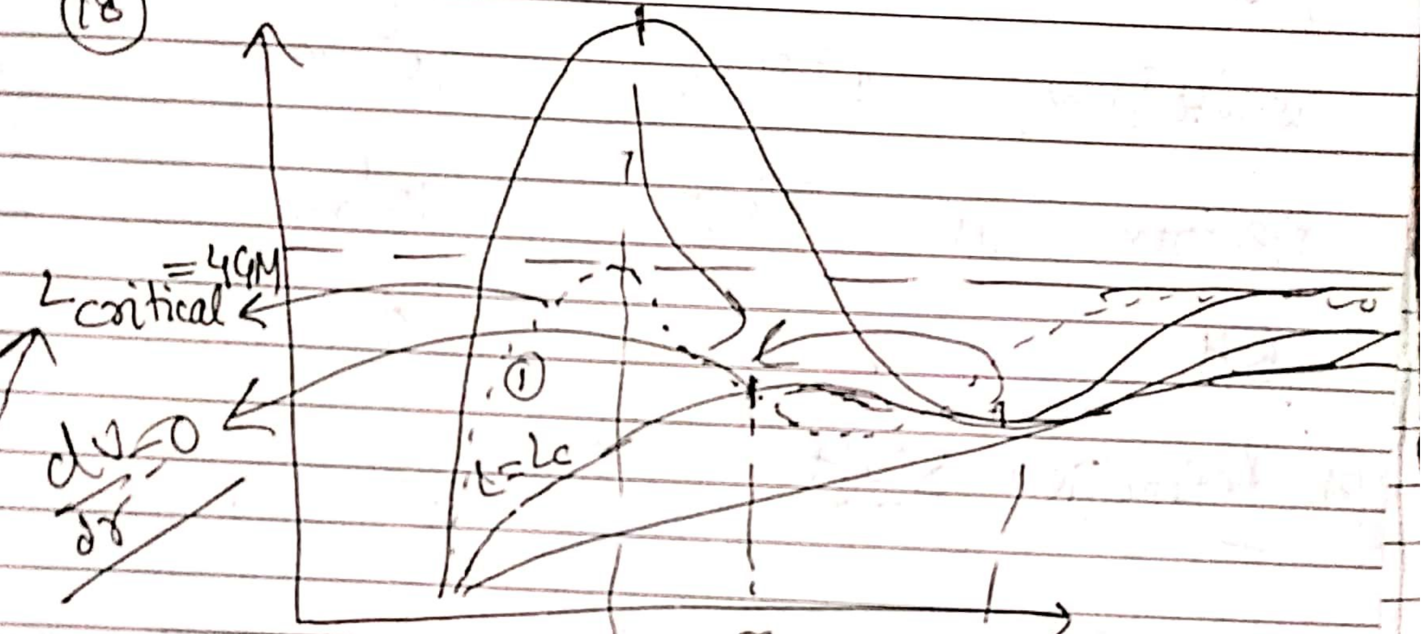
if L too low no chance of Circular Orbit

if L high then stable & Unstable Circular Orbit

(18)



$r_{\pm}$ for (1) $= 4GM, 12GM$

with $r < r_c$ No Stable orbits.

Rec.

after $L > L_c$

Unstable would be $r < r_c$

& Stable would be $r > r_c$.

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}$$

r_+ stable

r_- unstable

When $L^4 = 12G^2M^2L^2$

$\Rightarrow L^2 = 12G^2M^2$

When $r_+ = r_- = r_c$ = Critical Radius

at radius smaller than r_c No Stable Satellite.

$\boxed{r_c = 6GM}$

$= 2r_s$

$r_c = \frac{L^2}{2GM} = \frac{12G^2M^2}{2GM} = 6GM$

(23) in GR

$$r(\phi) = \frac{L^2}{GM\left[1+e\cos((1-\alpha)\phi)\right]}$$

$$\alpha = \frac{3G^2M^2}{L^2}$$

Periodic in $\frac{2\pi}{1-\alpha} = 2\pi(1+\alpha)$

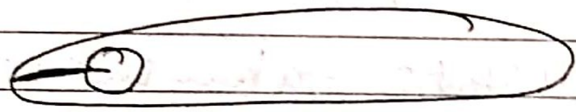
$$= 2\pi + \underbrace{2\pi\alpha}_{\Delta\phi}$$

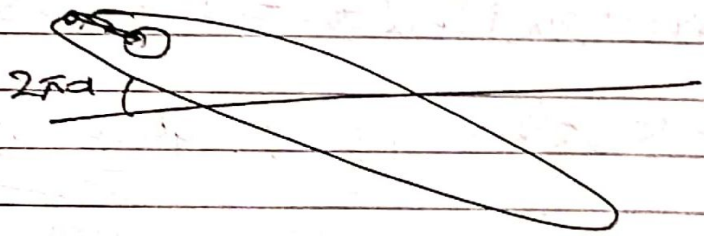
(24) Let Start from r_p

$\therefore$ coming to same position $r(\phi) = r_p$

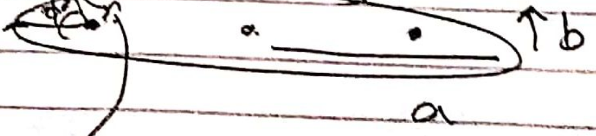
~~Angle Periodicity moves~~

Angle Changes by $2\pi\alpha$





(25) $\Delta\phi = \frac{6G^2M^2\pi}{L^2}$



Newton

$\frac{L^2}{GM} = (1-e^2)a$ $\quad r(\phi) = \frac{(1-e^2)a}{1+e\cos\phi}$ $\quad e = \sqrt{1-\frac{b^2}{a^2}}$

$\hookrightarrow L = \sqrt{GM(1-e^2)a}$

(26) in GR also

$$\frac{L^2}{GM} = (1-e^2)\,a$$

$$\therefore \Delta\phi = \frac{6\pi GM}{(1-e^2)\,a}$$

for small a $\Delta\phi \uparrow\uparrow$

↓

semimajor axis

↓

Mercury

(27) Predicted $\Delta\phi = 43$ arcsec/century

Observed $\Delta\phi = 5601$ arcsec/cent.

Assumptions Sun is static taken into acc.

Known correction Newton $= \Delta\phi = 5558$ arcs/cent

$5601 - 5558 = 43$ ∴ GR is correct

L-20

① $ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$

exp. Bec. $A, B, \geq 0$

EFE

Stationary

$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2$

$(\alpha = -\beta ?)$

② Inv. under $t \to -t$.

Stationary + $t \to -t$ Inv = Static

$$\boxed{e^{2\alpha} = 1 + \frac{c}{r}}$$

$c = -\frac{2GM}{r}$

③ Schwarzchild coordinates

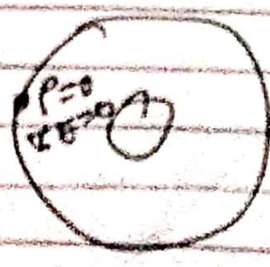
There are other coordinates which we will use but geometry would be same Schwarzchild.

$r = 2GM$ Schwarzchild Radius

④ $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

Sph. sym. $\vec{E} = f(r)\hat{r}$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow$



$$\nabla_i A^i = \frac{\partial_i\left(\sqrt{-g}\,A^i\right)}{\sqrt{-g}}$$

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$$r^4 \sin^2\theta \Rightarrow \sqrt{g} = r^2 \sin\theta$$

$$\frac{\partial_i\left(r^2 \sin\theta\, E^i\right)}{r^2 \sin\theta} \Rightarrow \frac{2}{r}E^r + \partial_r E^r$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r)\right) = 0$$

$$f(r) = \frac{\text{const}}{r^2}$$

$$\int \nabla\cdot\vec{E}\, d^3x = \int \vec{E}\cdot d\vec{a}$$

$$= \int f(r)\,\hat{r}\, d\theta\, d\phi$$

$$= E\, 4\pi r^2 = \int \frac{\rho}{\epsilon_0} d^3x = \frac{Q_{enc}}{\epsilon_0}$$

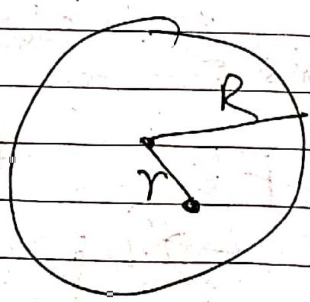
$$\boxed{E = \frac{Q}{4\pi r^2 \epsilon_0}}$$ outer Region

(5) Interior Region

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

Assume $\rho(r) = \begin{cases} \rho & r \le R \\ 0 & r \ge R \end{cases}$

$$Q_{tot} = \rho\,\frac{4}{3}\pi R^3$$



(4) $$\int \vec{\nabla}\cdot\vec{E}\, d^3x = \int \frac{\rho}{\epsilon_0} d^3x$$

$$4\pi r^2 = \int \vec{E}\cdot d\vec{a} = \frac{\rho}{\epsilon_0}\int d^3x = \frac{\rho}{\epsilon_0} 4\pi r^3 = \frac{Q_{tot}\, r^3}{\epsilon_0 R^3}$$

$$\boxed{E = \frac{Q_{tot}\, r}{4\pi\epsilon_0 R^3}}$$

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$$\boxed{\begin{aligned} E_{in} &\propto r \\ E_{out} &\propto \frac{1}{r^2} \end{aligned}}$$

$$\boxed{E_{in}(R) = E_{out}}$$

(6) Int. Schwarchild Sol^n

$T^{\mu\nu} \neq 0$

Assume Sph. symmetry

$\therefore$ Still $ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$

is still valid

in vaccum outer EFE (Birkhoff theorem)

But in Interior Assume t-independence

$\swarrow$

we have to show it is consistent

$\therefore$ $ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$

EFE

$\hookrightarrow G_{ij}$ is Diag

$\downarrow$

$G_{tt}\ G_{rr}\ G_{\theta\theta}\ G_{\phi\phi}$

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(7) Assume
Perfect fluid source.

$$T_{\mu\nu} = (p+\rho)\, U_\mu U_\nu + p\, g_{\mu\nu}$$

Let fluid be at rest in our frame

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dx^0}{d\tau}, 0, 0, 0\right)$$

in any coordinate

$$\therefore \quad g^{00} U_0 U_0 = g^{\mu\nu} U_\mu U_\nu = U^\mu U_\mu = -1$$

$$-e^{2\alpha} U_0 U_0 = -1$$

$$\boxed{U_0 = e^\alpha}$$

$$T^{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2\sin^2\theta\, p \end{pmatrix}$$

$\longrightarrow$ Get 4 Diff. Eq^n

$M(R) = M$

(1) Let $e^{2\beta(r)} = \left[1 - \frac{2GM(r)}{r}\right]^{-1}$

$$G_{tt} = 8\pi G\, T_{tt} \Rightarrow \underbrace{r^{-2}\left[e^{2(\alpha-\beta)}\left(-1+e^{2\beta}+2r\beta'\right)\right]}_{8\pi G e^{2\alpha}\rho}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \Big\}\ \text{in } M(r)$$

$$M(r) = 4\pi \int_0^r \rho(r)\, r^2\, dr \Rightarrow M(r=R) = M$$

(2) $G_{rr} = 8\pi G T_{rr}$ $\qquad \alpha' = \frac{d\alpha}{dr}$

$\beta \to M(r)$

$$\alpha' = \frac{GM(r) + 4\pi G r^3 p}{r(r - 2GM(r))}$$

(6) Using $\nabla_\alpha T^{\alpha\beta} = 0$
$\beta = r$

$$(p+\rho)\frac{d\alpha}{dr} = -\frac{dp}{dr}$$

G_{rr}

(3) $$\frac{dp}{dr} = -\frac{(\rho+p)\left[GM(r) + 4\pi G r^3 p\right]}{r\left[r - 2GM(r)\right]}$$ → Top man oppenheimer Volkoff Eqn

Start with
$\rho(r)$ given

We have assumed time Ind. Spher. sym.
we have assum perfect flud & we are in its rest frame.

a) Use (1) & get $M(r) \to \beta$

b) & now use (3) to get p

c) use (2) to get α

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L-22

(1) Mass is trying to pull Everything to the centre but p is pushing it away.

in $M > \frac{4R}{9G}$ mass is so much pulling it that p can't balance it off.

(6) $\therefore$ It will collapse & R will keep on decreasing $\therefore$ ~~mass will increase~~

for ρ const $\Big\{ M > \frac{4R}{9G}$ will be satisfied forever

(2) But is ρ = const a real model
~~Yes this is~~ good approx. → What about our time frame dep assumption?

But even when ρ(r)
By Buchdahl's Theorem $M > \frac{4R}{9G}$ still holds.

(3) Sun: $c^2 \frac{4}{9} \frac{4}{M} = 10^{27}$ kg

$M_\odot = 10^{30}$ kg

$\therefore$ It should collapse
But it doesn't coz of Nuclear R^n forces outside

But if Sun goes out of fuel then it should collapse?
No, these are EM, ~~weak~~ forces?

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Example

(8)

assume const ρ density

$$\rho(r) = \begin{cases} \rho_c = \text{const} & r < R \\ 0 & r > R \end{cases}$$

a) $$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho & r \le R \\ \frac{4}{3}\pi R^3 \rho & r > R \end{cases}$$

$$e^{2\beta(r)} = \left[1 - \frac{8\pi G\rho r^2}{3}\right]^{-1}$$

b) $$\frac{dp}{dr} = -\frac{(\rho+p)\left[G\frac{4}{3}\pi r^3\rho + 4\pi G r^3 p\right]}{r(r - 2G\,4/3\,\pi r^3\rho)}$$

$$\Rightarrow p(r) = \rho\left(\frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}}\right) \quad r<R$$

c) $e^{\alpha(r)}$

$$R^3 = 9R^2(R-2GM)$$
$$R^3 = 9R^3 - 18GM$$
$$18GM = \frac{4}{9}\frac{R^3}{G}$$

① $p(r) \uparrow$ with $r \downarrow$

② The pressure at $r=0$ Blows

~~Assumption~~ as $M \to \frac{4}{9}\frac{R}{G}$ ~~wrong time Ind wrong~~

if $M < \frac{4}{9}\frac{R}{G}$ DO then p is not ∞

But if $M \geq \frac{4}{9}\frac{R}{G}$ then $p \to \infty$

So this physically can't happen

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But Most Macroscopic obj. are Neutral
$\therefore$ No effect of EM.

Post fuel Burnout:

a) $E^{\ominus}$ degeneracy Pressure : White Dwarf.

$M < 1.4 M_{\odot}$

if $M > 1.4 M_{\odot}$ then it will continue to collapse

b) Neutron degeneracy pressure: Neutron Star

$M < 3 \sim 4 M_{\odot}$

c) if $M > 4 M_{\odot}$: Schwarzschild BH

4) $\to$ Sph. sym. EFE vacc (um)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$r \to 2GM$ for most astroph. obj

$r = 2GM < R$.

But Not for BH

5) if I am at $r = 30GM$ then BH & Star grav. effects are same. No diffrence.

Bec. geometry of BH & Star is same if assuming mass is same of both.

⑥ Escape velocity

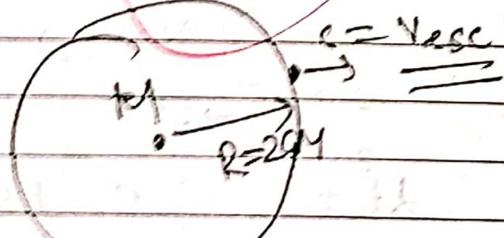
$$E_{tot} = \frac{mv^2}{2} - \frac{GMm}{R} = 0$$

$v = 0$

$$V_{esc} = \sqrt{\frac{2GM}{r}}$$

if $r = 2GM$
$v_{esc} = 1 = c$

⇒ Doubt



⑦ $$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$r \to (2GM, 0)$.

Coordinate Singularity
Not
⇒ a Curvature Singular

consider $\mathbb{R}^2$ (r, θ)

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

at $r = 0$ metric is fishy But pt. is fine

a) metric is coord. dep ⇒ metric is not good thing to look
⇓
check Invariance of Curvature
⇓
$R_{ijkl} = 0$ (as flat spacetime) ; $R^i_i = 0$, $R_{ijkl}R^{ijkl} = 0$

b) There are better coordinates to work with.

$\mathbb{R}^2$, $g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (x, y) Cartesian

In Curvature ~~Inv.~~ Singularity $R^i_{jkl} = \infty$

(8) in Schwarzchild $(t\ r\ \theta\ \phi)$

$r = 2GM$ → Sch. geom. in Sch. coordinates

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$R_{ij} = 0 \Rightarrow R = 0$

$$R^{ijkl}R_{ijkl} = \frac{48G^2M^2}{r^2} \qquad R_{ijkl} \neq 0$$

All other Inv. out of Riemann Tensor is 0

$\therefore$ $r = 2GM$ This inv. of $R^{ijkl}R_{ijkl}$ is all fine

$\therefore$ $r = 2GM$ is Coord. Singularity

But

$r = 0$ is True Curv. Singularity.

$\therefore$ Look for Better coordinates.

Sch. geom. in Other Coordinates

$(t\ r\ \theta\ \phi)_{\text{Sch.}} \longrightarrow (v\ r\ \theta\ \phi)$ Eddington Finkelstein

$$v = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right|$$

$$t = v - r - 2GM \ln\left|\frac{r}{2GM} - 1\right|$$

(Sch. geom. in EF coord.)

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$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

(v same unit as Dist.)

Solving this Eqn we get

$t = \pm\left[r + 2GM \ln\left(\frac{r}{2GM} - 1\right)\right]$ ← r^*

tortoise coord.

$v = t + r^*$

(9) $r = 2GM$ is all fine

$r = 0$ is Bad.

∴ Can't correct Curv. ~~Sing~~. Sing. By Coord.

(10)

Sch coord ✗

✓ Sch coord B

✓ Sch coord A

$r = 2GM$

As $r \to r_s$, $r^* \to -\infty$

$\frac{dr}{dr^*} = 1 - \frac{2GM}{r}$

$r \to r_s \Rightarrow \frac{dr}{dr^*} \to 0$

Going from A → B crossing $r = 2GM$ Sch. coord. are not gonna work. as at $r = 2GM$ Sch. coord. doesn't work

(11) Light Cones in Sch. Metric

$ds^2 = 0$

Not Straight line

Consider Only in radial motion

Sch. $0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$

EF $0 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr$

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

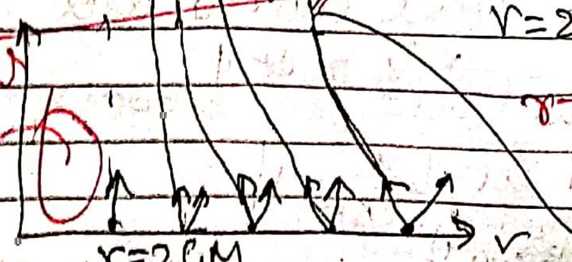
$r \to \infty$ $\frac{dt}{dr} = \pm 1$

$r = 2GM$ $\frac{dt}{dr} = \infty$ (Straight line)

$r \to 2GM$ all going asympt.

Sch

What happens here?



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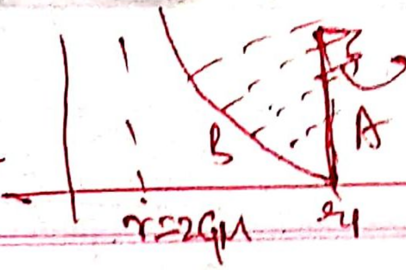
in Sch coord



at 45° | 2 observers A & B |

$d\tau_A, d\tau_B$ proper time of A & B

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$$d\tau_A^2 = \left(1 - \frac{2GM}{r_1}\right) dt^2 \qquad d\tau_B^2 = \left(1 - \frac{2GM}{r_2}\right) dt^2$$

$$\frac{\Delta\tau_A}{\Delta\tau_B} = 1 + GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

(12) EF

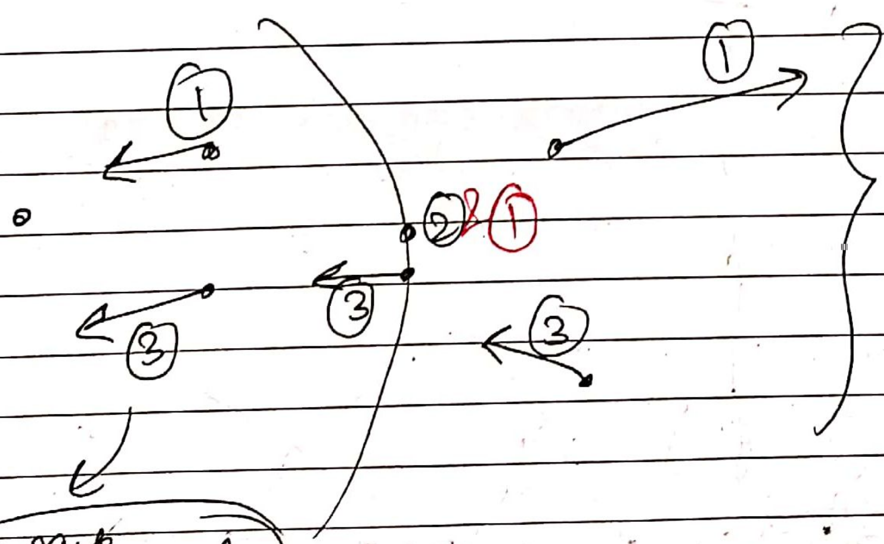
(1) $\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2GM}{r}\right) = \begin{cases} > 0 & r > 2GM \\ < 0 & r < 2GM \\ 0 & r = 2GM \end{cases}$

(2) $dr = 0$ & $r = 2GM$.

(3) $dv = 0 \Rightarrow v = \text{const}$

$t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right| = \text{const}$

$t \uparrow \quad r \downarrow$



for light

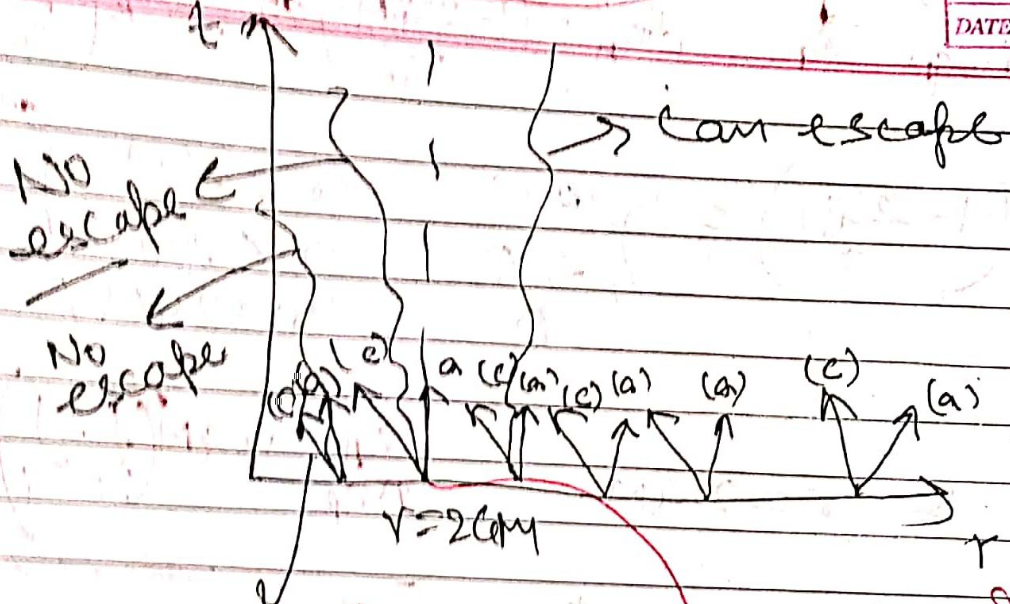
always come inside $r = 2GM$

Event horizon

Light & not anything else can come out from $r < 2GM$

$r > 2GM$

$\hat{t} = v - r$

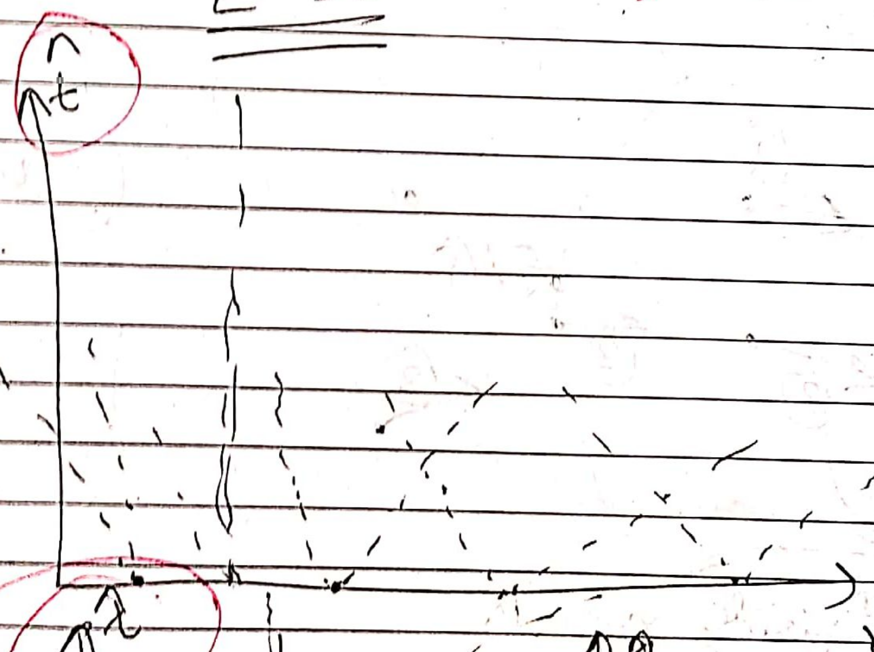


Both inside

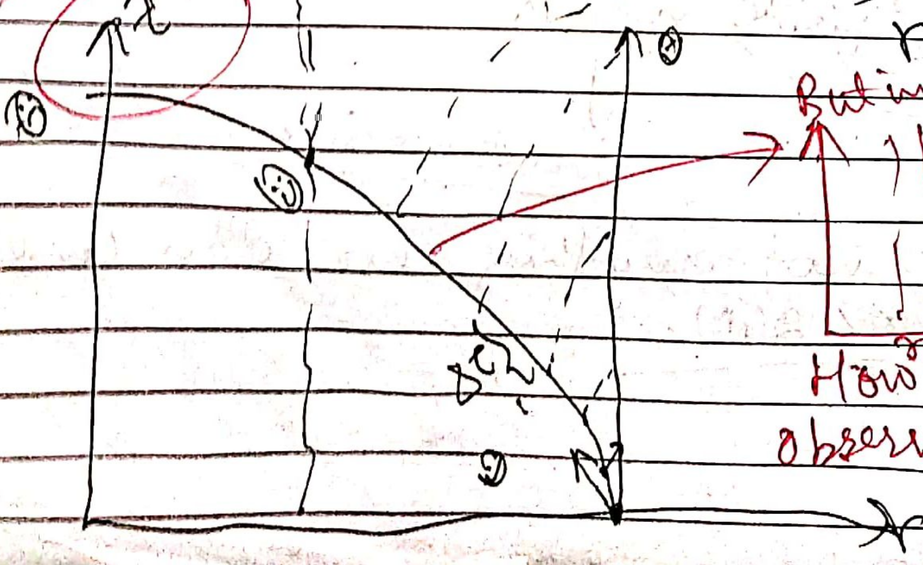
What would I see in locally inertial frame?

L-23

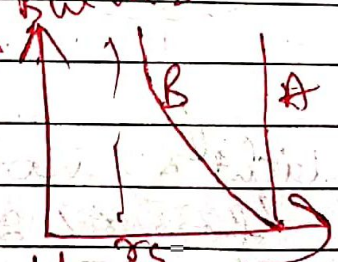
(1)



(2)



But in Sch coord



How to describe observer $r < r_s$ in Sch. coord

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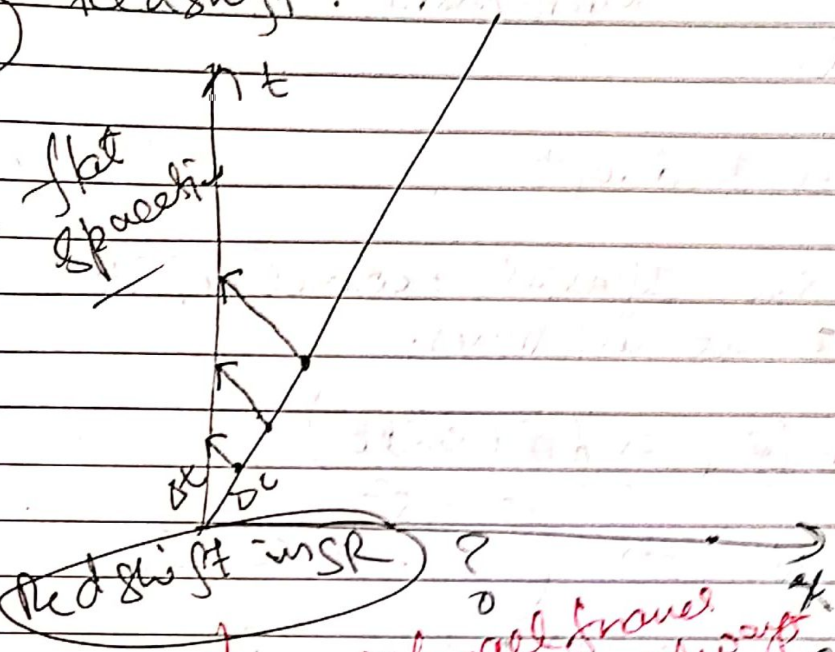
(3) Redshift

$\frac{d}{c} = t_1$

$\frac{2d}{c} = t_2$

$\Delta t = \frac{d}{c}$ → const.



~~X~~ c is const for all frames ∴ $\omega_1 = \omega_2$ always

But in Sch. geom. Δt keeps ↑

frequency keeps decreasing

"Redshifts"

(4) friend goes into BH in finite time Interval
But from my experience
they asymptotically reach BH & never pass through.
→ Redshifts
∴ fades away

(5) This Redshift is due to Motion + Geometry.

(6) $ds^2 = -dt^2 + dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$

b^2 → const.

$t \in (-\infty, \infty)$
$r \in [0, \infty)$
$\theta \in [0, \pi]$
$\phi \in [0, 2\pi]$

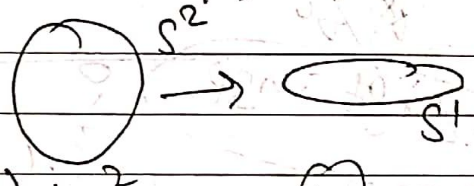
Not t dep + Not t dx } Static

→ for $r \to \infty$ Minkowski
→ S^2 - foliated.

as Metric is ind. of t.

freeze t & see spatial geometry, it would be same at all times.

$$ds^2 = dr^2 + (r^2+b^2)(d\theta^2 + \sin^2 d\phi^2)$$

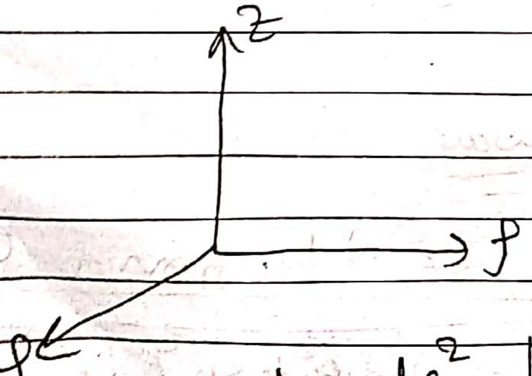
let $\theta = \pi/2 \Rightarrow$ 

$$ds^2 = dr^2 + (r^2+b^2)\, d\phi^2 \quad (1)$$

(II) $\mathbb{R}^3$ - $\{z, \rho, \varphi\}$ cylindrical.

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

To embed (1) in (II) we need $z(r,\phi)$ $\rho(r,\phi)$ $\varphi(r,\phi)$



let $\varphi = \phi$ (alligning)

$\therefore$ $z(r,\phi) = z(r)$
$\rho(r,\phi) = \rho(r)$

$$\therefore ds^2 = \left(\frac{\partial z}{\partial r}\right) dr^2 + \left(\frac{\partial \rho}{\partial r}\right) dr^2 + \rho^2 d\phi^2 \quad (2)$$

Comparing (1) & (2)

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial \rho}{\partial r}\right)^2 = 1$$

$$\rho^2 = r^2 + b^2$$

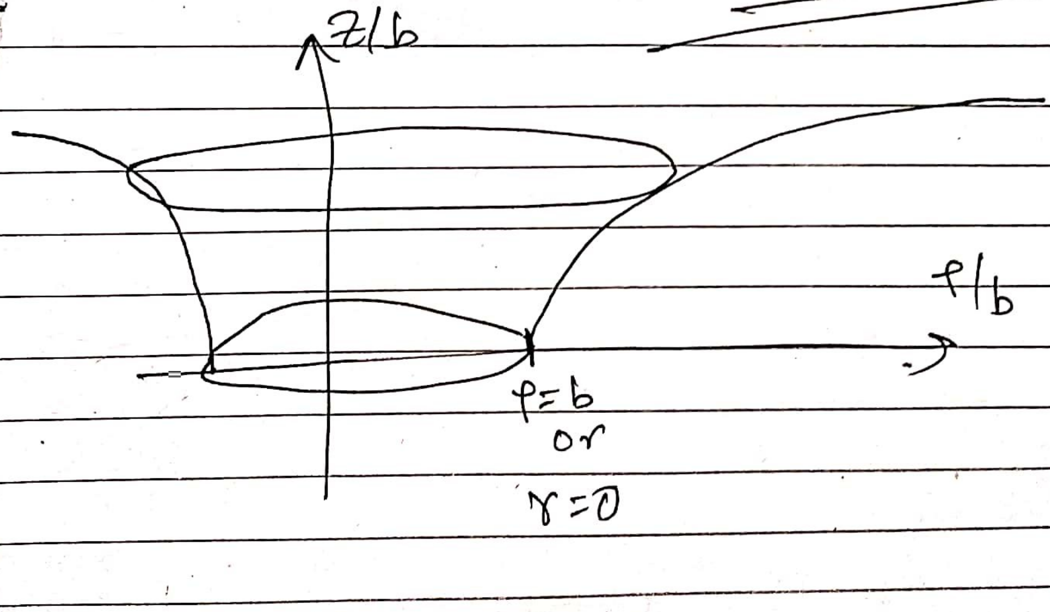
$$\frac{\partial \rho}{\partial r} = \frac{r}{\sqrt{r^2+b^2}}$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{r^2}{r^2+b^2} = 1 \Rightarrow z(r) = b\sinh^{-1}\left(\frac{r}{b}\right)$$

where $z(0)=0$

$$z(\rho) = b\sinh^{-1}\left(\sqrt{\frac{\rho^2}{b^2} - 1}\right)$$

for $r>0$ $z>0$



FRW Cosmology

① We are in the Universe ∴ we have to use $T_{\mu\nu} \neq 0$

Not like Sch. where we use $T_{\mu\nu} = 0$

② Cannot assume t ind.

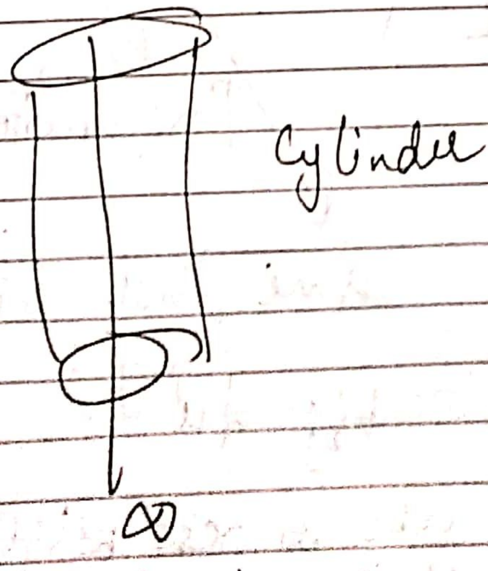
Einsten wanted to assume t ind. But not

③ Identify Symmetries

1. Symmetries

Spatial Symmetry ⇒ Homogenity
Spatial Symetry ⇒ Isotropy.

Homogenous But Not isotropic.



Isotropic But Not Homog.

Spherical Sym $\hat{r}$ $\hat{r}$

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(2) When Both Combined

We can go any place & look there would be isotropy (sph. sym).

∴ No centre of the Universe

(3) Combining Both we get maximally symmetric spatial geometric.

(4) Coordinates

"Comoving spatial Coordinate"

?

(5) $ds^2 = -dt^2 + R^2(t)\gamma_{ij}(u)\, du^i du^j$

$[L^2]$

$i, j = 1, 2, 3$

[Dimensionless]

time Ind. Spatial geom.

Spatial $\{ d\sigma^2 = \gamma_{ij}\, du^i du^j$

(6) Assume we are at rest with overall fluid of the Universe

Similar where we assume We are at Rest w.r.t Perf. fluid

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(7) Only spatial part is Max. Sym:

$R_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$

Only spatial comp.

$R_{jl} = \gamma^{ik} R_{ijkl} = 2K\gamma_{jl}$

$R = \gamma^{jl} R_{jl} = 6K$

How this comes?

(8) $\mathbb{R}^3$ $k=0$ $d\sigma^2 =$ flat $= d\chi^2 + \chi^2 d\Omega^2$

S^3 $k>0$ positive curvature $d\sigma^2 = d\chi^2 + \sin^2\chi\, d\Omega^2$

↳ 3-sphere (Closed spatial geom) $d\Omega_2^2$

H^3 $k<0$ $-ve$ $d\sigma^2 = d\chi^2 + \sinh^2\chi\, d\Omega^2$

↓ hyperbolic (Open spatial geom)

(9) $d\chi = \frac{dr}{\sqrt{1-kr^2}} \Rightarrow d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$

$k = 0, \pm 1$

(10) $ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$

Robertson - Walker

(11) Define $a(t) = \frac{R(t)}{R_0}$ (Dimensionless)

R_0 fixed length

$r \equiv R_0 \bar{r}$ Dimensful Radial coor

$\bar{K} = \frac{K}{R_0^2}$ Scale factor Dimensful Spatial Curv.

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

Metric is invariant under $a \to \lambda a$, $r \to r/\lambda$, $K \to \lambda^2 K$

2 Unknown $a(t)$, K

Metric reasonable form

$$ds^2 = -dt^2 + a^2[d\rho^2 + S_K^2(\rho)\, d\Omega^2]$$

Instead of finding 10 comp. we need to find just 2 here.

(12) 3 sphere

$$d\sigma^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$0 \le \chi \le \pi$ $(x\,y\,z) \to (\theta, \phi)$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$$dV = J\, d\chi\, d\theta\, d\phi$$

$$dV = \sqrt{g_{3sp.}}\, d\chi\, d\theta\, d\phi$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sqrt{g}\, d\chi\, d\theta\, d\phi$$

$= 2\pi^2$ Finite volume

(13) Hyperbloid

$$d\sigma^2 = d\chi^2 + \sinh^2\chi\,(d^2\theta + \sin^2\theta\, d\phi^2)$$

$0 \le \chi \le \infty$
$0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$

Infinite volum

2sphere Area $= \int \sqrt{g}\, d\theta\, d\phi$

χ const $= \sinh\chi \int \sin\theta\, d\theta \int d\phi$

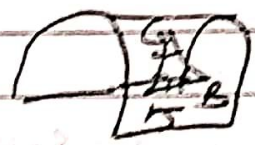
$= 4\pi \sinh\chi$

Increases with χ

(14) Tringle Angles

Sphere $\angle A + \angle B + \angle C > 180^\circ$

flat $\angle A + \angle B + \angle C = 180$



Hyperbolic $\angle A + \angle B + \angle C < 180^\circ$.

(15) $g_{ij} \longrightarrow g_{ij} + \underbrace{\epsilon g_{ij}}_{\delta g_{ij}}$

(16) $d\eta = \frac{dt}{a} \Rightarrow ds^2 = a^2(\eta)\left[-d\eta^2 + dx^2 + x^2 d\Omega^2\right]$

$\underline{\underline{k=0}}$

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17) Physical coord r^i are related to comoving x^i

$$r^i = a(t)\, x^i$$

$$V^i_{phys} = a(t)\frac{dx^i}{dt} + \frac{da}{dt}\frac{r^i}{a}$$

$$V^i_{ph} = V^i_{pec.} + Hx^i$$

$\downarrow$ Hubble flow

Hubble parameter $= \frac{\dot{a}}{a}$

18) $l(t)$: Proper Distance.
l_0 : spatial Dist.

$$l(t) = a(t)\, l_0$$

Observers are fixed at their resp. coordinates
$\Rightarrow$ Comoving length doesn't change.

Let comoving Distance B/w 2 obs δx

$$\delta l = a\, \delta x$$

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Physical velocity

(19) $\frac{d\delta l}{dt} = \delta v = \frac{d}{dt}(a\,\delta x)$

$= \dot{a}\,\delta x = \frac{\dot{a}}{a}\,\delta l$

Hubble Law $\boxed{\delta v = H\,\delta l}$

(20) $\delta t = c\,\delta l$

(1) ν
(2) $\nu + \delta\nu$ } Doppler shift

$\frac{\delta\nu}{\nu} = -\frac{\delta v}{c} = -\frac{\delta a}{a}$ → HOW $-\frac{\dot{a}}{a}\frac{\delta l}{c} = -\frac{\dot{a}\,\delta t}{a} = -\frac{\delta a}{a}$

$\boxed{\nu(t)\,a(t) = \text{const}}$ — (I)

$\boxed{\text{if } a\uparrow \quad \lambda\uparrow}$

(21) for Exp. Univ. $a(t)\uparrow$

let $1+z = \frac{a_0}{a(t)}$ a_0 at present time

$\boxed{z = \frac{\Delta\lambda}{\lambda}} = -\frac{\Delta\nu}{\nu} = \frac{\delta v}{c}$

(22)

$dt^2 = a^2\,dx^2$ Incoming

$dt = \pm a\,dx \Rightarrow dt = -a\,dx$

t_2: time light is recieved
t_1: light emitted time $x = -\int_{t_2} \frac{dt}{a(t)} = \int_{t_1}^{t_2} \frac{dt}{a}$

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Let Next Crest emitted at $t_1 + \delta t_1$

$$\delta t_1 = \frac{1}{\nu_1}$$

This will reach $x = 0$ at $t_2 + \delta t_2$

x is comoving cord: fixed

$$\boxed{dx = 0}$$

$$\int_{t_1}^{t} \frac{dt}{a} = \int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{dt}{a}$$

possible only if $\dfrac{\delta t_2}{a(t_2)} = \dfrac{\delta t_1}{a(t_1)}$

$$\frac{\delta t_2}{\delta t_1} = \frac{\nu_1}{\nu_2} \quad \text{using (1)}$$

$$1+z = \frac{\nu_1}{\nu_2} = \frac{a(t_2)}{a(t_1)} = \frac{a_0}{a(t)}$$

(23) Massive ν Eq of motion $\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a}$

$\Downarrow$

I find $\rightarrow E^2 = p^2c^2 + m^2c^4$

~~comoving fixed $\Rightarrow p^i = 0$~~

$p^0 = E/c$

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(24) $\frac{d\lambda}{\lambda} = \frac{dv}{c} = dz$.

$dt = \frac{dl}{c}$. $dt = \frac{da}{Ha}$

$\dot{a} = Ha \longrightarrow dz \frac{d\lambda}{\lambda} = H dt = \frac{da}{a}$

$$\boxed{\begin{aligned} \frac{d\lambda}{\lambda} &= \frac{da}{a}. \\ \lambda &= c\,a. \\ \lambda(a) &= \lambda_{obs}\, a \end{aligned}}$$

$\lambda_{obs} \downarrow$ Today

(25) $\longrightarrow \frac{\lambda_{obs}}{\lambda_e} = \frac{\lambda_{obs} - \lambda_e}{\lambda_e} + 1$

$\Rightarrow \rho \propto \frac{1}{a}$

$= \frac{\Delta\lambda}{\lambda_e} + 1$

$\because = z + 1 = \frac{a_0}{a(t)}$

Let $a(t_0) = 1$

$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \dot{a}(t_0)$

$\Rightarrow$