$u^i \nabla_i u^j = 0$ (GR) $\rightarrow$ affine. Vector is // Transported.

affine $\leftarrow$ $u^i \partial_i u^j = 0$ (SR)
Rate of change of components $= 0$ along the curve
$\therefore$ Straight line
In any coord. system (t, x) (t, r, θ, ϕ)

$$df = f(x^i + dx^i) - f(x^i)$$
$$= \partial_i f \, dx^i$$

$$\frac{df}{d\lambda} = \partial_i f \frac{dx^i}{d\lambda} = \partial_i f \, u^i$$

or

$$d\rho = \rho(t+dt, x+dx) - \rho(t, x)$$
$$= \frac{\partial \rho}{\partial t} dt + \frac{\partial \rho}{\partial x} dx$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\left(\frac{dx}{dt}\right)$$

in Non Relativistic limit.

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho$$

$$J^i = (\rho, \rho \vec{v})$$

$$\partial_i J^i = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = \frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v}$$

$$\therefore \quad \partial_i J^i = \frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v}$$

Continuity Eqn $\underline{\underline{\partial_i J^i = 0}}$

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① Definition of vector?

② Definition of Dual vector?
(without coordinate & with coordinate Transfn)

without coordinate Def: $\boxed{\text{Dual vector is the linear operation on a vector to produce } \mathbb{R}.}$

Rate of change of scalar fn along the curve.

③ As this operation produces $\mathbb{R}$ which is scalar $\therefore$ $\partial_{i'} f\, u^{i'} = \partial_i f\, u^i$

$\therefore \left\{\frac{df}{d\lambda}\right. = \partial_i f u^i = \partial_{i'} f u^{i'}$ is invariant

④ Vectors & Dual vectors arise from the Differentiation of scalar function.

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\lambda} = f_{,i}\, u^i = \partial_i f\, u^i$$

where λ is the parameter used to parameterize the curve γ.

$\partial_i f$: Dual vector, gradient of fn.

u^i : vector tangent to curve γ

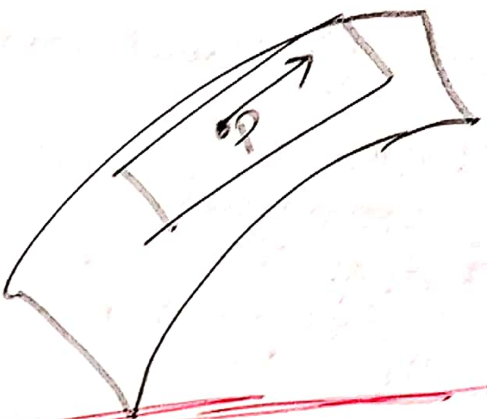
⑤ Invariance of $\frac{df}{d\lambda}$ can also be seen Explicitly Calculating.

$$\left.\begin{aligned} \partial_{i'} f &= \partial_i f\, \partial_{i'} x^i \\ u^{i'} &= \frac{dx^{i'}}{d\lambda} = \partial_i x^{i'} u^i \end{aligned}\right\} \Rightarrow \partial_{i'} f\, u^{i'} = \partial_i f\, u^i$$

⑥ Scalar Means Invariant under Coord. Transfn.

⑦ Definition of Tensor ?
→ Linear operation on vectors & dual forms which produces $\mathbb{R}$. eg. $T^{\alpha}_{\beta} v^{\beta} p_{\alpha} = \#$

$$\rightarrow T^{\alpha'\beta'} = \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} \left.\frac{\partial x^{\beta'}}{\partial x^{\beta}}\right|_{P} T^{\alpha\beta}$$



Definition of vector ?

$$V^{\alpha'} = \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} V^{\alpha}$$

Why Tensor Multiplication, Add^n, Sub^n, Contrac^n not defined?

⑧ Imp Tensors & vectors are not defined on manifold but defined on Tangent space at pt. P.

⑨ V. Imp Addition of Tensor at diff. Points is not a Tensorial qty.
∴ Integeration of Tensors is not valid describe

$$T^{\alpha'}(P) + T^{\alpha'}(Q) = \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} T^{\alpha}(P) + \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{Q} T^{\alpha}(Q)$$

$$\neq \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} \left(T^{\alpha}(P) + T^{\alpha'}(Q)\right)$$

⑩ V. Imp Multiplication of Tensors at diff. Points is not a Tensorial qty.

$$T^{\alpha'}(P)\, T^{\beta'}(Q) = \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} \left.\frac{\partial x^{\beta'}}{\partial x^{\beta}}\right|_{Q} T^{\alpha}(P)\, \& \, T^{\beta}(Q)$$

$$\neq \left.\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|_{P} \left.\frac{\partial x^{\beta'}}{\partial x^{\beta}}\right|_{P} T^{\alpha}(P)\, T^{\beta}(Q)$$

(11) Th. As addition & Multiplication at different points is not a Tensorial qty ∴ Contraction is also not a Tensorial qty. (3)

V. Imp

Proof: $T^{\alpha'\beta'}(P)\, T_{\beta'\gamma'}(Q) = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\Big|_P \frac{\partial x^{\beta'}}{\partial x^{\beta}}\Big|_P \frac{\partial x^{\beta}}{\partial x^{\beta'}}\Big|_Q \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Big|_Q T^{\alpha\beta}(P)\, T_{\beta\gamma}(Q)$

$\neq \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\Big|_P \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Big|_P T^{\alpha}_{\gamma}(P)$

What are 3 Conseq of Defining g_{ij} on M — pseudo Riemannian

metric Tensor on manifold makes it a metric space & then we are able to calculate distances in spacetime

(12) g_{ij} is the 2nd Rank symmetric Tensor which also represents the gravitational field.
(metric Tensor is used to define inner product b/w 2 vectors $g(\vec{A},\vec{B}) = g_{\alpha\beta}A^{\alpha}B^{\beta}$)

② Causality ③ Gravity Information

Def: $ds^2 = g_{ij}\, dx^i dx^j$

(13) Only $\partial_i f$ is a Tensorial qty.

$\partial_i A^{\alpha}$ is not a Tensorial qty

Why $\partial_i f$ is Tensor But not $\partial_i A^{\alpha}$ on Manifold

What is the Def. of Covariant Derivative? Sudipta L-1

$Q: x + dx$

$P: x$

(14) $dA^{\alpha} \equiv A^{\alpha}(Q) - A^{\alpha}(P)$

$= \partial_{\beta} A^{\alpha}\, dx^{\beta}$

is not a Tensor qty.

2 ways to check dA^{α} is not a tensor ① $A^{\alpha}(Q) - A^{\alpha}(P)$ ② $\partial_{\beta}A^{\alpha}$ is not Tensor

$DA^{\alpha} = A^{\alpha}_{T}(P) - A^{\alpha}(P)$ is a Tensor qty

$= dA^{\alpha} + \delta A^{\alpha}$ where δA^{α} not Tensor

$= A^{\alpha}(Q) - [A^{\alpha}(P) - \delta A^{\alpha}]$

$\delta A^{\alpha} \equiv A^{\alpha}_{T}(P) - A^{\alpha}(Q)$

Demanding δA^{α} linear in both A^{μ} & dx^{β}

$\delta A^{\alpha} = \Gamma^{\alpha}_{\mu\beta} A^{\mu} dx^{\beta}$ $\Gamma^{\alpha}_{\mu\beta}$: Connection

→ Not a Tensor But $\delta\Gamma$ is a Tensor.

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$$\therefore \frac{DA^\alpha}{d\lambda} = U^\beta \nabla_\beta A^\alpha$$

where $\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\beta\gamma} A^\gamma$.

(15) Definition: Parallel Transport.

A tensor field $T^{\alpha\beta\gamma}{}_{\mu\dots}$ is said to be // transport along the curve γ if its covariant derivative along the curve vanishes : $0 = \frac{DT^{\alpha\beta\gamma}{}_{\mu\dots}}{d\lambda} = U^\gamma \nabla_\gamma T^{\alpha\beta\gamma\dots}{}_{\mu\dots}$

which implies that

$$\frac{DA^\alpha}{d\lambda} = \frac{A^\alpha_T(P) - A^\alpha(P)}{d\lambda} = 0$$

Def:
Covariant Derivative along the Curve
$\frac{DA^\alpha}{d\lambda} = \nabla_i A^\alpha u^i$
Covariant Derivative all along Space
$\nabla_i A^\alpha = \partial_i A^\alpha + \Gamma A$

$$A^\alpha_T(P) = A^\alpha(P)$$

$\therefore$ vector Remains same along the curve.

$\therefore$ // Transported.

(16) It can be shown from Def. of // Transp. that constant vectors in flat spacetime in Cartesian Coordinate are // Transported.

$$U^\gamma \nabla_\gamma A^\alpha = \frac{dA^\alpha}{d\lambda} = 0$$

Constant vector
i.e. Components Independent

(17) As $\frac{DA^\alpha}{D\lambda}$ is a Tensor $\therefore$ $\nabla_\beta A^\alpha$ is a Tensor

Earlier in Paddy:

By Action Principle we got

$$\underbrace{U^\beta}_{\text{Tensor}} \nabla_\beta u^\alpha = \underbrace{0}_{\text{Scalar}}$$

$\therefore$ Tensor

(18) $\boxed{\text{Transformation of the Connection}}$

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\gamma\beta} A^\gamma$$

$$\Gamma^\alpha_{\gamma\beta} A^\gamma = \underline{\nabla_\beta A^\alpha - \partial_\beta A^\alpha}$$

Transf. these RHS & get Transf. of $\Gamma^{\alpha'}_{\gamma'\beta'}$

Earlier in Paddy:

By Action Principle

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\mu}}{2}\left(-\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta}\right)$$

Now we know $g^{\alpha\beta}$ is Tensor & its Transf^n

$\therefore$ we get Transf^n of $\Gamma^{\alpha'}_{\beta'\gamma'}$.

(19) Demanding (1) D obeys Product Rule
(2) $D = d$ for scalar.
Need to Define the Transfor for Dual vector & Tensor?
(20) We can derive $D(P_\alpha)$ with the above we demand

$D(A^i A_i) \neq 0$
$Df \neq 0$
But $D(u^i u_i) = $
as $u^i u_i = 1$ and
But $A^i A_i$ depend on x

$$D P_\alpha = (\nabla_\mu P_\alpha)\, dx^\mu$$

$$\nabla_\mu P_\alpha = \partial_\mu P_\alpha - \Gamma^\gamma_{\mu\alpha} P_\gamma$$

(21) Similarly from (19) Demand we can get result of covariant derivative of Tensors

By $D(T^{\alpha\beta}) = D(V^\alpha V^\beta)$ we get $DT^{\alpha\beta} = \nabla_\mu T^{\alpha\beta} dx^\mu$

$$\nabla_\mu T^{\alpha\beta} = \partial_\mu T^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma} T^{\gamma\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\gamma}$$

(22)

Earlier in Paddy:

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\mu}}{2}\left(-\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta}\right)$$

→ As is $g_{\alpha\beta}$ is symmetric : $\Gamma^\alpha_{\beta\gamma}$ is sym in β & γ

→ Also Γ is metric compatible → identity

(23) But in our approach

connection is completely arbitrary

$$\delta A^\alpha = \Gamma^\alpha_{\beta\gamma} A^\beta dx^\gamma$$

(24) Earlier : $\nabla_i g_{ab} = 0$ from $\nabla_i V^\alpha = g_{\alpha\beta} \nabla_i V^\beta$

or By Explicit Calculation $\nabla_i T_{ab}$

To Prove

If D follows Product Rule then ∇ also?

Proof: $D(A^\alpha B^\beta) = B^\beta D A^\alpha + A^\alpha D B^\beta$

$= B^\beta u^r \nabla_r A^\alpha + A^\alpha u^r \nabla_r B^\beta$

$u^r (\nabla_r (A^\alpha B^\beta)) = u^r (B^\beta \nabla_r A^\alpha + A^\alpha \nabla_r B^\beta)$

$\nabla_b \phi = \partial_b \phi$

$\boxed{\text{Also } D = d \text{ for scalar} \Leftrightarrow \nabla = \partial \text{ for scalar}}$

(2) To Prove

If ∇ follows PR then D also?

Proof: $D(A^\alpha B^\beta) = U^r \nabla_r (A^\alpha B^\beta) = B^\beta u^r \nabla_r A^\alpha + A^\alpha u^r \nabla_r B^\beta$

$= B^\beta D A^\alpha + A^\alpha D B^\beta$

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25 But in our approach
Demanding Γ is symmetric ~~& metric compatible~~

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} \quad \cancel{\& \ \nabla_\gamma g_{\alpha\beta} = 0}$$

26 From these Demands we can show $\Gamma^{\rho}_{\beta\alpha} = \frac{g^{\rho\gamma}}{2}(-\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta})$

Proof: $\nabla_\gamma g_{\alpha\beta} = 0 \Rightarrow \partial_\gamma g_{\alpha\beta} = \Gamma^{\mu}_{\gamma\alpha} g_{\mu\beta} + \Gamma^{\mu}_{\gamma\beta} g_{\mu\alpha}$

$$① \begin{cases} \partial_\gamma g_{\alpha\beta} - \Gamma^{\mu}_{\gamma\beta} g_{\mu\alpha} = \Gamma^{\mu}_{\gamma\alpha} g_{\mu\beta} \\ \partial_\alpha g_{\gamma\beta} - \Gamma^{\mu}_{\alpha\beta} g_{\mu\gamma} = \Gamma^{\mu}_{\gamma\alpha} g_{\mu\beta} \end{cases}$$

$$② \begin{cases} \partial_\gamma g_{\alpha\beta} - \Gamma^{\mu}_{\gamma\alpha} g_{\mu\beta} = \Gamma^{\mu}_{\gamma\beta} g_{\mu\alpha} \\ \partial_\beta g_{\alpha\gamma} - \Gamma^{\mu}_{\beta\alpha} g_{\mu\gamma} = \Gamma^{\mu}_{\gamma\beta} g_{\mu\alpha} \end{cases}$$

See L-1 19

Adding all above we get out our eqn

27 Definition: Geodesic is any curve which extremizes the distance B/w fixed points.

Let λ be arbitrary parameter
Distance B/w P & Q is given by

$$l = \int_P^Q \sqrt{\pm g_{ij} \dot{x}^i \dot{x}^j} \, d\lambda$$

$$\boxed{A = -ml = -m \int_P^Q \sqrt{\pm g_{ij} \dot{x}^i \dot{x}^j} \, d\lambda}$$

Action of free particle in GR?

where $\dot{x}^i = \frac{dx^i}{d\lambda}$

+ : if the curve is Timelike
− : if the curve is Spacelike

It is assumed curve is nowhere null.

(8)

(28) By action Principle

let $L = \sqrt{\pm g_{ij} \dot{x}^i \dot{x}^j}$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

$$\Rightarrow \pm \frac{d}{d\lambda}\left(\frac{g_{ij}\dot{x}^i \delta^i_\alpha}{\sqrt{\pm g_{ij}\dot{x}^i\dot{x}^j}}\right) = \pm \frac{\partial_\alpha g_{ij} \dot{x}^i \dot{x}^j}{2\sqrt{\pm g_{ij}\dot{x}^i\dot{x}^j}}$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{g_{ij}\dot{x}^i\delta^i_\alpha}{L}\right) = \frac{\partial_\alpha g_{ij}}{2L}\dot{x}^i\dot{x}^j$$

$$= \frac{1}{L}\frac{d}{d\lambda}\left(g_{ij}\dot{x}^i\delta^i_\alpha\right) + g_{ij}\dot{x}^i\delta^i_\alpha \frac{d}{d\lambda}\left(\frac{1}{L}\right)$$

$$= \frac{\partial_\alpha g_{ij}}{2L}\dot{x}^i\dot{x}^j$$

$$= \frac{d}{d\lambda}\left(g_{j\alpha}\dot{x}^\alpha\right) + g_{j\alpha}\dot{x}^\alpha\left(L\frac{d}{d\lambda}\left(\frac{1}{L}\right)\right) = \frac{\partial_\alpha g_{ij}\dot{x}^i\dot{x}^j}{2}$$

$$= \frac{d\dot{x}^j}{d\lambda} + \dot{x}_j \quad L\frac{d}{d\lambda}\left(\frac{1}{L}\right) = \frac{\partial_\alpha g_{ij}\dot{x}^i\dot{x}^j}{2}$$

$$= \frac{d\dot{x}^j}{d\lambda} + \Gamma^j_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\dot{x}^j L\frac{d}{d\lambda}\left(\frac{1}{L}\right)$$

$$\frac{d(\ln L)}{d\lambda} = \frac{1}{L}\frac{dL}{d\lambda}$$

$$\frac{d(\ln L^{-1})}{d\lambda} = -\frac{d\ln L}{d\lambda} = -\frac{1}{L}\frac{dL}{d\lambda} = L\frac{d(1/L)}{d\lambda}$$

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(9)

$$\therefore \boxed{u^i \nabla_i u^j = u^j \frac{d(\ln L)}{d\lambda}}$$

Let $k(\lambda) = \dfrac{d \ln L}{d\lambda}$

$$\therefore \boxed{u^i \nabla_i u^j = k(\lambda)\, u^j}$$

(29) $k(\lambda) = \dfrac{d \ln L}{d\lambda} = \dfrac{1}{L}\dfrac{dL}{d\lambda}$

Earlier in Paddy:

$u^i \nabla_i u^j = 0$

$u^i = \dfrac{dx^i}{d\tau}$ Let $\tau = f(\lambda)$

$\dfrac{d}{d\tau} = \dfrac{\partial\lambda}{\partial f}\dfrac{d}{d\lambda}$

$\Rightarrow u^b \nabla_b u^a = \dfrac{f''}{f'} u^a = g(\lambda)\, u^a$

(30) Proper Time τ : $d\tau^2 = g_{ij}\, dx^i dx^j$

Proper Distance s : $ds^2 = -g_{ij}\, dx^i dx^j$

(31) $L = \sqrt{g_{ij}\dfrac{dx^i}{d\tau}\dfrac{dx^j}{d\tau}}$ for Timelike

$\boxed{\text{Affine Parameter}}$

$= \dfrac{d\tau}{d\tau} = 1$

$L = \sqrt{-g_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}}$ for spacelike

$= \dfrac{ds}{ds} = 1$

32 When Geodesic is Timelike $dx^i dx_i > 0$

Parameter used is Proper time τ

When Geodesic is spacelike $dx^i dx_i < 0$

Parameter used is Proper Dist. s.

33 As from 31

$L = 1 \Rightarrow k = 0 \Rightarrow u^a \nabla_a u^b = 0$

$\Rightarrow$ Tangent vector is // Transported along geodesic.

34 Earlier By Reparameterization

$\tau = f(\lambda)$

we got $u^b \nabla_b u^a = \frac{f''}{f'} u^a$ $\quad f' = \frac{\partial f}{\partial \lambda}$

$\boxed{L = \sqrt{g_{ij} \frac{dx^i dx^j}{d(a\tau) d(a\tau)}} \quad L = \frac{1}{a} \Rightarrow k = 0}$

$\therefore$ if f is Linear in λ
we get same Geod. Eqn

$\therefore$ $\left.\begin{array}{l} \lambda = a\tau + b \\ \lambda = as + b \end{array}\right\}$ Affine Parameters

35 Practical method for Calculating Γ.

$A = \int d\lambda \left(g_{ij} u^i u^j \right) = \int L \, d\lambda$

Putting in Euler Lag. Eqn
we get $u^b \nabla_b u^a = 0$

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This is the quickest way to get Christofel symbol.

$L_1 = g_{ij} u^i u^j$ — L_1 gives geodesic Eqn only if λ is Affine.

$L_2 = \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$

~~$u^\alpha \nabla_\alpha u^\beta = k(\lambda) u^\beta$ (λ is the parameter)~~

~~where $u^\alpha = \frac{dx^\alpha}{d\lambda}$~~

~~Now let $\lambda \rightarrow \lambda' = a\lambda + b$~~

~~$k(\lambda) = \frac{d \ln L}{d\lambda'}$ $\quad L = \sqrt{g_{ij} \frac{dx^i}{d\lambda'} \frac{dx^j}{d\lambda'}}$~~

~~As A is Reparameterizable $\therefore \lambda \rightarrow \lambda' = a\lambda + b$ will not change eqn $u^\alpha \nabla_\alpha u^\beta = k(\lambda') u^\beta$~~

~~if $\lambda = \tau / s \Rightarrow u^\alpha \nabla_\alpha u^\beta = 0$~~

$$K.E. = \sum_{i,k=1}^{N} a_{ik}(q)\, \dot{q}^k \dot{q}^i$$

$N =$ Dof.

(36) Let λ be affine Parameter (12)

$$u^i = \frac{dx^i}{d\lambda}$$

then along affinely parameterized geodesic, the scalar qty $\epsilon = u^i u_i$ will be constant.
i.e. if the geodesic is Timelike at one pt. it will Remain timelike

$$\frac{d\epsilon}{d\lambda} = \frac{d(u^i u_i)}{d\lambda} = \frac{D(u^i u_i)}{d\lambda} = u^j \nabla_j (u^i u_i)$$

$$= \underbrace{\underbrace{u^j u^i \nabla_j u_i} + \underbrace{u^j u_i \nabla_j u^i}}_{\text{Geod. Eqn}}$$

$$\frac{d\epsilon}{d\lambda} = 0$$

$$\underline{\underline{\epsilon = \text{Const}}}$$

(37) Usefulness of Non affine Parameterization;

(1) Non affine Parameters can be used to easy the calculation Ex. in Expanding Universe metric L-(2)

(2) Non affine parameters can be used to describe Null geodesic
as affine parameters can't be used for them & Diff Eq can't be solved
as $d\tau = 0$
L-(2) (27)

(38) For parallel Transport we need Christoffel Symbol /connection defined on M. (13)

There is another way of transporting a vector field along the curve which doesn't require the knowledge of Γ but only requires the knowledge of vector field defined on whole spacetime (atleast neighbourhood of γ curve). In // Transport we need vector field only along the curve.

(39) $\frac{du_i}{d\tau} = \frac{\partial_i g_{kj}}{2} u^k u^j$

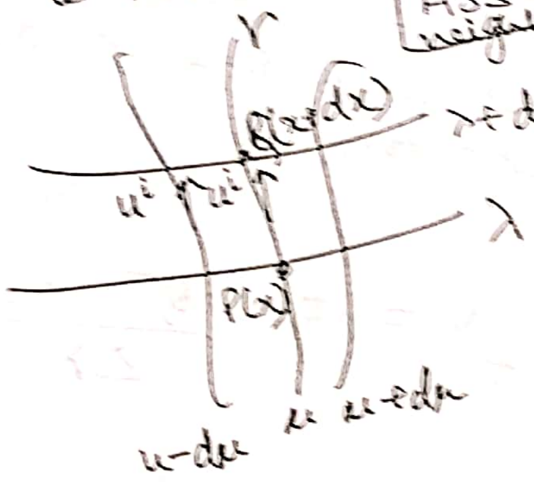
if metric is ind. of x^l then P_l is conserved

But due to the freedom of choice of coordinate systems, g_{ab} maybe ind. of x^l in one coordinate system but, but depends on x^l in new coordinate system.

(40) $\therefore$ Metric is independent of a particular coordinate is not a covariant statement.

$\boxed{\text{Assuming } u^i \text{ vector field } \exists \text{ in the neighbourhood of curve}}$

(41)



μ specifies the curve we are considering.

λ specifies the position we are on the curve.

$x^{i'} = x^i + dx^i = x^i + u^i d\lambda$

This is the infinitesimal coordinate transformation to a new set of primed coordinates.

The vector field v^a changes to:

$v^{i'}(x') = \frac{\partial x^{i'}}{\partial x^j} v^j(x)$ $\longrightarrow$ $v^{i'}(Q) = \frac{\partial x^{i'}}{\partial x^j}\Big|_Q v^j(P)$

$= v^i(x) + d\lambda\, v^j(x)\, \partial_j u^i$

Such an infinitesimal sliding can be treated as coordinate transformation, changing components of any tensorial objects by usual rules. Second, it will attribute different coordinate labels to the events in spacetime.

$\therefore\ x^{i'} = x^i + u^i d\lambda = x^i + dx^i$ can be interpreted as coordinates of Q.

$$\therefore\ v^{i'}(Q) = v^i(P) + d\lambda\, v^j(P)\ \partial_j u^i$$

But

$$v^i(Q) = v^i(x+dx) = v^i(P) + d\lambda\, u^j \partial_j v^i(P).$$

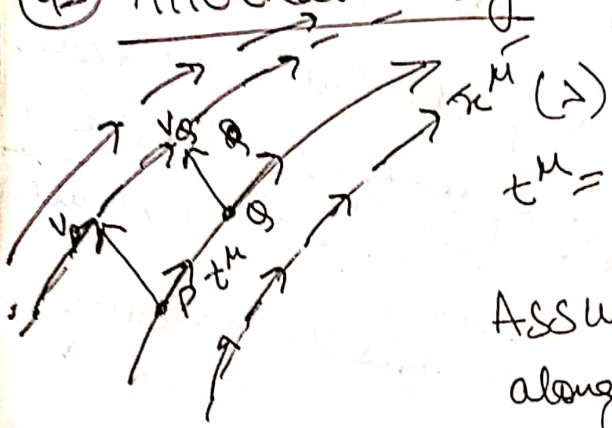
$$\mathcal{L}_u v^i(P) \equiv \lim_{d\lambda \to 0} \frac{v^i(Q) - v^{i'}(Q)}{d\lambda}$$

$$= u^j \partial_j v^i - v^j \partial_j u^i = u^j \nabla_j v^i - v^j \nabla_j u^i$$

= covariant vector. ↳ see (63)

Ch- ↳ why do we need vector field in spacetime

(42) Alternate Way



$$t^\mu = \frac{\partial x^\mu}{\partial \lambda}$$

Assuming t^μ is not only defined along the worldline But also around the neighborhood.

Integral Curves

$$t^\mu = \frac{\partial x^\mu}{\partial \lambda}$$

$$\mathcal{L}_t V = \lim \frac{V_Q - V_{P\to Q}}{\Delta\lambda}$$

Depending on the Initial Condition we get diff. Integral Curves.

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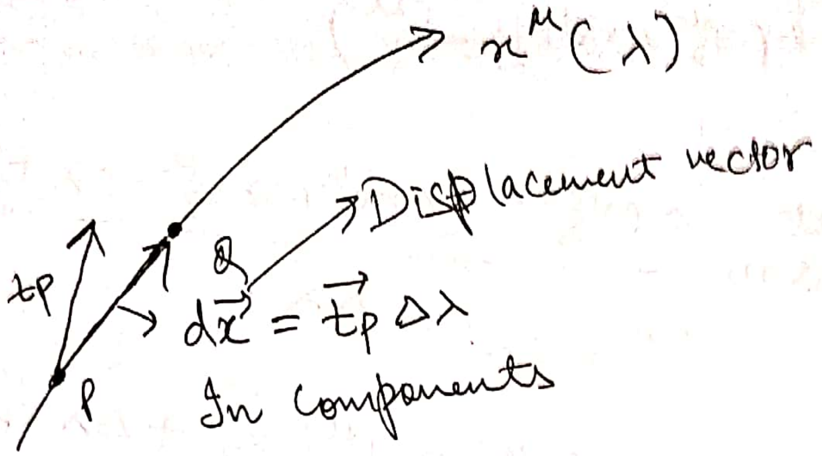
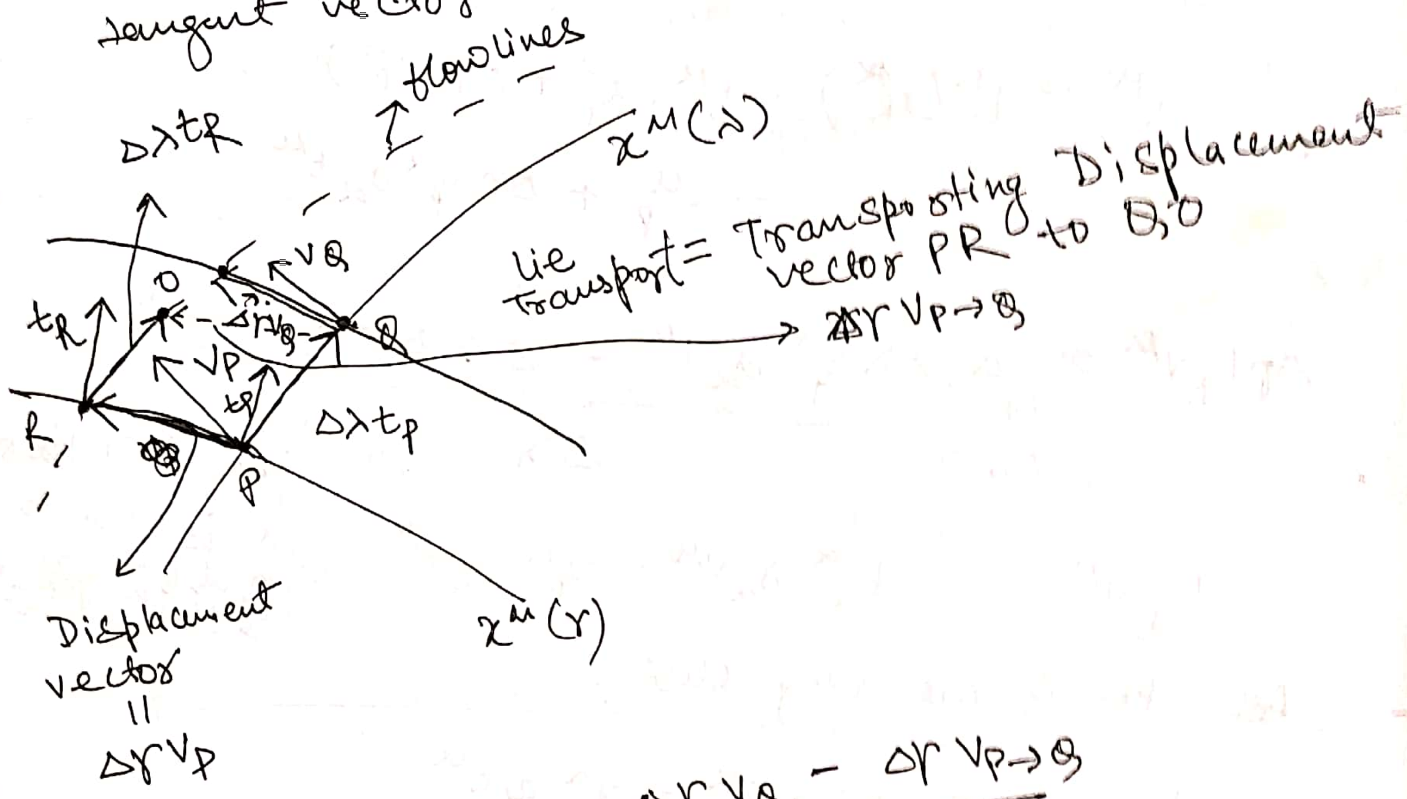
(13) (15)



In components

$$x^\mu_Q - x^\mu_P = \Delta x^\mu = t^\mu \Delta\lambda$$

(14) Let us take another curve with V^μ as its tangent vector



(15) $$\Delta\gamma \mathcal{L}_t V = \lim \frac{\Delta\gamma V_Q - \Delta\gamma V_{P\to Q}}{\Delta\lambda}$$

But we know for any vector A

$$\Delta A^\mu = A^\mu_f - A^\mu_i$$

$$\Delta\gamma V^\mu_{P\to Q} = x^\mu_O - x^\mu_Q$$

$$= (x^\mu_P + \Delta\gamma V^\mu_P + \Delta\lambda t^\mu_R) - (x^\mu_P + \Delta\lambda t^\mu_P)$$

$$\Delta\gamma\, v^{\mu}_{P\to Q} = \Delta\gamma\, v^{\mu}_{P} + \Delta\lambda\left(t^{\mu}_{R} - t^{\mu}_{P}\right)$$

$$\Delta\gamma\, L_t v^{\mu} = \lim_{\Delta\lambda\to 0} \frac{\Delta\gamma\, v^{\mu}_{Q} - \left(\Delta\gamma\, v^{\mu}_{P} + \Delta\lambda\, t^{\mu}_{R} - \Delta\lambda\, t^{\mu}_{P}\right)}{\Delta\lambda}$$

$$v^{\mu}_{Q} = v^{\mu}(Q) = v^{\mu}(x^{\mu}_{Q}) = v^{\mu}(x^{\alpha}_{P} + \Delta\lambda\, t^{\alpha}_{P})$$
$$= v^{\mu}(x^{\alpha}_{P}) + \Delta\lambda\, t^{\alpha}_{P}\,\partial_{\alpha} v^{\mu}$$

$$t^{\mu}_{R} = t^{\mu}(x^{\alpha}_{R}) = t^{\mu}(x^{\alpha}_{P} + \Delta\gamma\, v^{\alpha}_{P})$$
$$= t^{\mu}_{P} + \Delta\gamma\, v^{\alpha}_{P}\,\partial_{\alpha} t^{\mu}$$

$$\Delta\gamma\, L_t v^{\mu} = \lim_{\Delta\lambda\to 0} \frac{\Delta\gamma\,\Delta\lambda\, t^{\alpha}_{P}\,\partial_{\alpha} v^{\mu} - \Delta\lambda\,\Delta\gamma\, v^{\alpha}_{P}\,\partial_{\alpha} t^{\mu}}{\Delta\lambda}$$

$$L_t v^{\mu} = t^{\alpha}_{P}\,\partial_{\alpha} v^{\mu} - v^{\alpha}_{P}\,\partial_{\alpha} t^{\mu}$$

Imp $\mathcal{L}_t v^{\mu}$ is Tensorial Object ① Due to $v_{P\to Q} - v_Q$ ② $t^{\alpha}\nabla_{\alpha} v^{\mu} - v^{\alpha}\nabla_{\alpha} t^{\mu}$

As P & Q are very close

$$\boxed{\begin{aligned} L_t v^{\mu} &= t^{\alpha}\,\partial_{\alpha} v^{\mu} - v^{\alpha}\,\partial_{\alpha} t^{\mu} \\ &= t^{\alpha}\nabla_{\alpha} v^{\mu} - v^{\alpha}\nabla_{\alpha} t^{\mu} \end{aligned}}$$

Vector

= Lie Derivative of vector field v ~~wrt vector~~ w.r.t. t vector field.

Imp $\boxed{\mathcal{L}_t v^{\mu} = -\mathcal{L}_v t^{\mu}}$

(46) Extending Lie Derivative to Tensor $\begin{pmatrix} n \\ m \end{pmatrix}$:

Tensorial Eqn if holds in flat local Inertial frame then it is true in general.

Proof $A^{\alpha\beta} \stackrel{*}{=} A^{\alpha}B^{\beta}$ (let)

$A^{\bar{\alpha}\bar{\beta}} = \partial_{\alpha} x^{\bar{\alpha}} \partial_{\beta} x^{\bar{\beta}} A^{\alpha\beta}_{\text{Inertial}} = A^{\bar{\alpha}} B^{\bar{\beta}}$

$\therefore$ holds in genl

Transfn from Inertial to Non Inertial.

(17)

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ = Scalar fn

Define $\mathcal{L}_t f \equiv \dfrac{df}{d\lambda} = t^{\alpha}\partial_{\alpha} f$

$\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ = covector

Nomen $w = w_{\mu} e^{\mu}$

$\uparrow$ component

$$\mathcal{L}_t(w_{\mu}V^{\mu}) = t^{\alpha}\partial_{\alpha}(w_{\mu}V^{\mu})$$

Demanding Lie Derivative Obeys Product Rule

$$V^{\mu}\mathcal{L}_t w_{\mu} + w_{\mu}\mathcal{L}_t V^{\mu} = t^{\alpha}V^{\mu}\partial_{\alpha}w_{\mu} + t^{\alpha}w_{\mu}\partial_{\alpha}V^{\mu}$$

$$V^{\mu}\mathcal{L}_t w_{\mu} + w_{\mu}(\cancel{t^{\alpha}\partial_{\alpha}V^{\mu}} - V^{\alpha}\partial_{\alpha}t^{\mu}) = t^{\alpha}V^{\mu}\partial_{\alpha}w_{\mu} + \cancel{t^{\alpha}w_{\mu}\partial_{\alpha}V^{\mu}}$$

$$V^{\mu}\mathcal{L}_t w_{\mu} = V^{\mu}(t^{\alpha}\partial_{\alpha}w_{\mu} + w_{\alpha}\partial_{\mu}t^{\alpha})$$

$$\boxed{\mathcal{L}_t w_{\mu} = t^{\alpha}\partial_{\alpha}w_{\mu} + w_{\alpha}\partial_{\mu}t^{\alpha}}$$

$\Rightarrow \begin{pmatrix} n \\ m \end{pmatrix} \equiv T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$ $\qquad \mathcal{L}_t(T^{ij}_{k} V^k w_i z_j) = t^{\alpha}\partial_{\alpha}(\ f\ )$

$$\boxed{\begin{aligned}\mathcal{L}_t T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} &= t^{\alpha}\partial_{\alpha} T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} - T^{\alpha\mu_2 \cdots \mu_n}_{\nu_1 \cdots \nu_m}\partial_{\alpha}t^{\mu_1} \\ &\quad - \cdots \\ &\quad + T^{\mu_1 \cdots \mu_n}_{\alpha\nu_2 \cdots \nu_m}\partial_{\nu_1}t^{\alpha} + \cdots\end{aligned}}$$

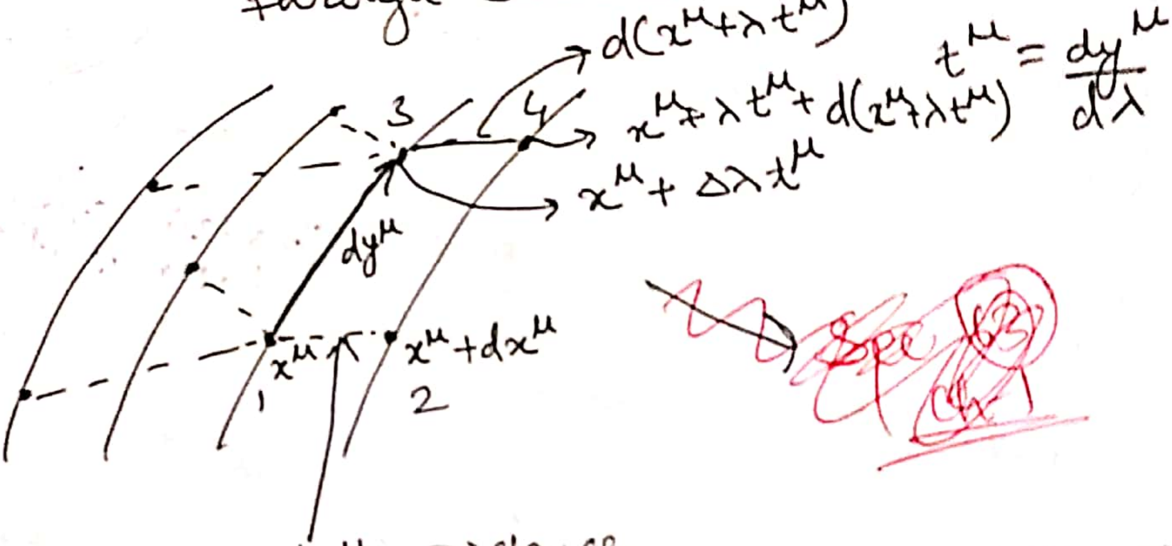
(47) $\mathcal{L}_t g_{ij} = t^\alpha \partial_\alpha g_{ij} + g_{\alpha j} \partial_i t^\alpha + g_{i\alpha} \partial_j t^\alpha = \nabla_i t_j + \nabla_j t_i$

Spacetime Symmetries

Physical Motivation for $\nabla_\alpha g_{\beta\gamma} = 0$ (10)

Let there be Congruence of curves.

Congruence : Set of curves s.t. one curve passing through each event



dx^μ = Distance

Def: Symmetry

(48) Spacetime is symmetric if network of distances is unchanged when the pts along the integral curves is unchanged. Euclidean $d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$
i.e. $dl^2 = dx^2 + dy^2 + dz^2$; $g_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$; $dl^2 = g_{ij} dx^i dx^j$

(49) ∴ if the spacetime is symmetric Difference B/W 2 point in a Manifold?

then $ds^2_{12} = ds^2_{34}$

$$g_{ij}(x^\mu)\, dx^i dx^j = g_{ij}(x^\mu + \Delta\lambda t^\mu)\, d(x^i + \Delta\lambda t^i)\, d(x^j + \Delta\lambda t^j)$$

(50) $$g_{ij}(x^\mu) dx^i dx^j = (g_{ij}(x^\mu) + (\partial_\alpha g_{ij}) t^\alpha d\lambda)$$
$$(dx^i + dx^\beta \partial_\beta t^i \Delta\lambda)$$
$$(dx^j + dx^\gamma \partial_\gamma t^j \Delta\lambda)$$

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$$(\partial_\alpha g^{ij}) \, t^\alpha \, d\lambda \, dx^i dx^j + g_{ij} \, dx^\beta \partial_\beta t^i \Delta\lambda \, dx^j$$
$$+ g_{ij} \, dx^r \partial_r t^j \Delta\lambda = 0$$

$$t^\alpha \partial_\alpha g_{ij} \cancel{dx^i dx^j} + g_{\beta j} \partial_i t^\beta + g_{ir} \partial_j t^r = 0$$

$$t^\alpha \partial_\alpha g_{ij} + g_{\beta j} \partial_i t^\beta + g_{ir} \partial_j t^r = 0$$

$$\mathcal{L}_t g_{ij} = 0$$

$$\boxed{\therefore \text{ Spacetime is symmetric along integral curve of vector field } K \text{ if } \mathcal{L}_K g_{ij} = 0 \qquad K = \text{killing vector field}}$$

(5) What if One of the coordinate basis is the killing vector ? Let x^0 be coord. Basis.

Let $K = \frac{\partial x}{\partial x^0} = (1, 0, 0, 0)$ where $K^i = \frac{\partial x^i}{\partial x^0} \left(t^i = \frac{\partial x^i}{\partial \lambda} \right)$

$$\mathcal{L}_K g_{ij} = K^\alpha \partial_\alpha g_{ij} + g_{\beta j} \partial_i K^\beta + g_{ir} \partial_j K^r = 0$$

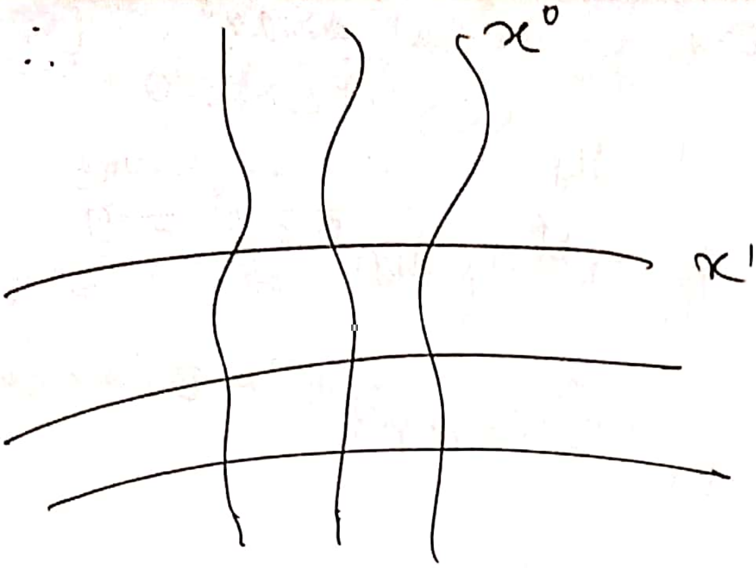
$$= \partial_0 g_{ij} + g_{0j} \cancel{\partial_i \frac{\partial x}{\partial x^0}} + g_{i0} \cancel{\partial_j \frac{\partial x}{\partial x^0}} = 0$$

$$= \partial_0 g_{ij} = 0$$

$$\frac{\partial g_{ij}}{\partial t} = 0$$

If metric Tensor is ind. of one of the coordinate say x^0, then $\frac{\partial x}{\partial x^0}$ is killing vector field.
$(1, 0, 0, 0)$

∴



The spacetime is symmetric under translations along x^0 coordinate lines.

(52)

(53) If g_{ij} is not ind. of any coord. it doesn't imply there is no symmetry.
Because still $\mathcal{L}_K g_{ij} = 0$ can be:
It is just that killing vector fields are not coordinate basis in that coordinate system.

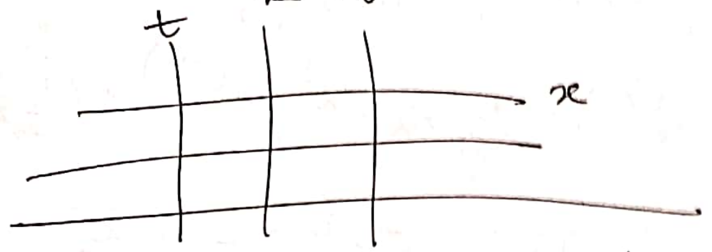
(54) Example (21)

$ds^2 = -e^{x} dt^2 + dx^2$

$g_{ij} = \begin{pmatrix} -e^{x} & 0 \\ 0 & 1 \end{pmatrix}$ Ind. of t $\equiv$ Time Translation Symmetry.

$k = \frac{\partial \vec{x}}{\partial t}$ = killing vector field

$\therefore \mathcal{L}_k g_{ij} = 0$



let $\left.\begin{array}{l} t = \bar{t} \\ x = \bar{t}\bar{x} \end{array}\right\}$ $\begin{array}{l} dt = d\bar{t} \\ dx = \bar{t}\, d\bar{x} + \bar{x}\, d\bar{t} \end{array}$

$dx^2 = \bar{t}^2 d\bar{x}^2 + \bar{x}^2 d\bar{t}^2 + 2\bar{t}\bar{x}\, d\bar{t}\, d\bar{x}$

$ds^2 = -e^{\bar{t}\bar{x}} d\bar{t}^2 + \bar{t}^2 d\bar{x}^2 + \bar{x}^2 d\bar{t}^2 + 2\bar{t}\bar{x}\, d\bar{t}\, d\bar{x}$

$= (\bar{x}^2 - e^{\bar{t}\bar{x}})\, d\bar{t}^2 + \bar{t}^2 d\bar{x}^2 + 2\bar{t}\bar{x}\, d\bar{t}\, d\bar{x}$

$g'_{ij} = \begin{pmatrix} \bar{x}^2 - e^{\bar{t}\bar{x}} & \bar{t}\bar{x} \\ \bar{t}\bar{x} & \bar{t}^2 \end{pmatrix}$ Not Ind of $\bar{t}$

But still the symmetry is there.

$k_1 = \frac{\partial \vec{x}}{\partial t} = \frac{\partial \bar{t}}{\partial t}\frac{\partial \vec{x}}{\partial \bar{t}} + \frac{\partial \bar{x}}{\partial t}\frac{\partial \vec{x}}{\partial \bar{x}}$

$= \frac{\partial \vec{x}}{\partial \bar{t}} - \frac{x}{t^2}\frac{\partial \vec{x}}{\partial \bar{x}}$

$= \frac{\partial \vec{x}}{\partial \bar{t}} - \frac{\bar{x}}{\bar{t}}\frac{\partial \vec{x}}{\partial \bar{x}}$ = killing vector field.

$\mathcal{L}_{k_1} g'_{ij} = 0$ $\quad k_1 \equiv k_1^{\alpha} = \left(1, -\frac{\bar{x}}{\bar{t}}\right)$

$$\mathcal{L}_{K_1} g'_{ij} = K_1^\alpha \partial_\alpha g'_{ij} + g'_{\alpha j}\partial_i K_1^\alpha + g'_{i\alpha}\partial_j K_1^\alpha$$

$$\mathcal{L}_{K_1} g'_{00} = K_1^\alpha \partial_\alpha g'_{00} + g'_{\alpha 0}\partial_0 K_1^\alpha + g'_{0\alpha}\partial_0 K_1^\alpha$$
$$= K_1^\alpha \partial_\alpha g'_{00} + 2 g'_{\alpha 0}\partial_0 K_1^\alpha$$
$$= \partial_0 g'_{00} - \frac{\bar{x}}{\bar{t}}\partial_{\bar{x}} g'_{00} + 2 g'_{x0}\partial_0\left(-\frac{\bar{x}}{\bar{t}}\right)$$
$$= \partial_0 g'_{00} - \frac{\bar{x}}{\bar{t}}\partial_{\bar{x}} g'_{00} + 2 g'_{x0}\frac{\bar{x}}{\bar{t}^2}$$
$$= \partial_{0'}(\bar{x}^2 - e^{\bar{t}\bar{x}}) - \frac{\bar{x}}{\bar{t}}\partial_{\bar{x}}(\bar{x}^2 - e^{\bar{t}\bar{x}}) + \frac{2\bar{x}\bar{t}}{\bar{t}^2}$$
$$= -\bar{x}e^{\bar{t}\bar{x}} - \frac{2\bar{x}^2}{\bar{t}} + \bar{x}e^{\bar{t}\bar{x}} + 2\frac{\bar{x}^2}{\bar{t}}$$
$$= 0$$

(55) $\mathcal{L}_K V^\alpha = K^i \partial_i V^\alpha - V^i \partial_i K^\alpha$
$$= K^i \nabla_i V^\alpha - V^i \nabla_i K^\alpha$$

$$\mathcal{L}_K V_\alpha = K^i \partial_i V_\alpha + V_i \partial_\alpha K^i$$
$$= K^i \nabla_i V_\alpha + V_i \nabla_\alpha K^i$$

metric compatible connection

But $\mathcal{L}_K g_{ij} = K^\alpha \partial_\alpha g_{ij} + g_{\alpha j}\partial_i K^\alpha + g_{i\alpha}\partial_j K^\alpha$

$= K^\alpha \cancelto{0}{\nabla_\alpha g_{ij}} + g_{\alpha j}\nabla_i K^\alpha + g_{i\alpha}\nabla_j K^\alpha = \nabla_i K_j + \nabla_j K_i$

from (50) spacetime is symmetric along integral curve of vector field K means $\mathcal{L}_K g^{ij} = 0$

i.e. It means K vector $\exists$ s.t.

$$\nabla^i K^j + \nabla^j K^i = 0$$

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(56) In general, if the Killing vector = (1,0,0,0) i.e. (23)
Integral curves are x^0, tangent vector $t^a = \delta^a_0$

$$\mathcal{L}_t T^{a---}_{b---} = t^i \partial_i T^{a----}_{b-----} + \cancel{\left(\text{Derivative of } t^i\right)}$$

$$= t^i \partial_i T^{a---}_{b---}$$

$$= \frac{\partial}{\partial x^0} T^{a---}_{b----}$$

(57) Proof

$$x^{i'} = x^i + dx^i = x^i + u^i d\lambda$$

$$g^{i'k'}(x^{l'}) = \partial_l x^{i'} \partial_m x^{k'} g^{lm}(x^l)$$

$$= (\delta^i_l + \partial_l u^i d\lambda)(\delta^k_m + \partial_m u^k d\lambda)(g^{lm})$$

$$= g^{ik}(x^l) + g^{im}\partial_m u^k d\lambda + g^{lk}\partial_l u^i d\lambda$$

$$g^{ik}(x^{l'}) = g^{ik}(Q) = g^{ik}_{\circ}(x^l + dx^l)$$

$$= g^{ik}(x^l) + d\lambda\, u^j \partial_j g^{ik}\Big|_P$$

$$\frac{g^{ik}(x^{l'}) - g^{i'k'}(x^{l'})}{d\lambda} = \mathcal{L}_K g^{ik}$$

$$= \frac{d\lambda\, u^j \partial_j g^{ik} - g^{im}\partial_m u^k d\lambda - g^{lk}\partial_l u^i d\lambda}{d\lambda}$$

$$\mathcal{L}_K g^{ik} = u^j \partial_j g^{ik} - g^{im}\partial_m u^k - g^{lk}\partial_l u^i$$

$$= \nabla^{i} K^j + \nabla^j K^i$$

$$x^{i'} = x^i + \xi^i$$

$$g^{i'k'}(x^{l'}) = g^{lm}(x^l)\, \partial_l x^{i'} \partial_m x^{k'}$$

$$= g^{ml}(\delta^i_l + \partial_l \xi^i)(\delta^k_m + \partial_m \xi^k)$$

$$= g^{ik}(x^l) + g^{im}\partial_m \xi^k + g^{kl}\partial_l \xi^i$$

$$g^{ik}(x^{l'}) = g^{ik}(x^l) + \xi^l \partial_l g^{ik}(x^l)$$

$$g^{ik}(x^{l'}) - g^{i'k'}(x^{l'})$$

$$= \xi^l \partial_l g^{ik}(x^l) - g^{im}\partial_m \xi^k - g^{kl}\partial_l \xi^i$$

$$= \delta g^{ik} = \mathcal{L}_t g^{ik}$$

$$= \nabla^i \xi^k + \nabla^k \xi^i$$

(58) But $\nabla_a T^{ab} = 0$ only if g_{ab} is ind. of some coordinate. (24)

If k_b is the killing vector & $\nabla_a T^{ab}=0$ is conserved then

$$\cancel{\frac{d(T^{ab}k_b)}{dx}} \quad \nabla_a (T^{ab} k_b) = \cancel{(\nabla_a T^{ab})} k_b + (\nabla_a k_b) T^{ab}$$

$$= \left(\frac{\nabla_a k_b + \nabla_b k_a}{2}\right) T^{ab}$$

$$= 0$$

$$\therefore T^{ab} k_b = p^a$$

$$\Rightarrow \nabla_a p^a = 0$$

(59) In the derivation of lie derivative, we need ~~g_{ab}~~ to be defined in the neighbourhood of γ ~~But A_i to be only defined on the γ.~~

(60) ~~To~~ Let A^α not depend on x^0 in one coord. syst.

then $\partial_0 A^\alpha \overset{*}{=} 0 \overset{*}{=} \partial_\beta A^\alpha u^\beta$

$$u^\alpha \overset{*}{=} (1,0,0,0)$$

$$\partial_\beta u^\alpha \overset{*}{=} 0$$

$\nabla_i k_j + \nabla_j k_i \overset{*}{=} 0$

But $\mathcal{L}_u A^\alpha = \partial_\beta A^\alpha u^\beta - \partial_\beta u^\alpha A^\beta$

$\partial_i x^{i'} \partial_j x^{j'} (\nabla_{i'} k_{j'} + \nabla_{j'} k_{i'}) = 0$ $\quad \overset{*}{=} 0$

$\therefore \nabla_{i'} k_{j'} + \nabla_{j'} k_{i'} = 0$ But as $\mathcal{L}_u A^\alpha$ is Tensor

$\therefore$ It is zero in all coord. system. $\mathcal{L}_u A^\alpha = 0$

Tensor zero in one frame then Tensor zero in all frames.

(61) Th: if in a given coordinate system a Tensor do not depend on particular x^i (coordinate basis) then the lie derivative of the tensor in the direction of u^α vanishes.

$u^\alpha \overset{*}{=} \delta^\alpha_i$ (in that part. coord-system)

$\boxed{\text{2 most Signf. Theorems.}}$

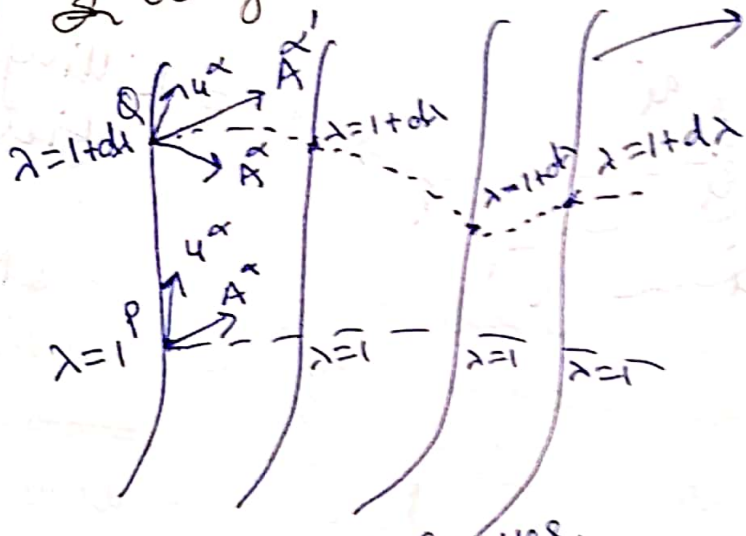
(62) Th: If $\mathcal{L}_u T^{\alpha ---}_{\beta ---} = 0$ then a coordinate system can be constructed such that

$u^a \overset{*}{=} \delta^a_i$ & $\partial_a T^{\alpha ---}_{\beta ---} \overset{*}{=} 0$.

$\mathcal{L}_u g^{ij} = 0$

(63) 2 ways of getting $\mathcal{L}_u g^{ij}$

(1)



u^α Integral Curves are not necessarily geodesics.

⇒ A^α tangent to cross curves.

⇒ We have independence of giving $\lambda=1$ to cross curve as $u^\alpha = \frac{dx^\alpha}{d\lambda}$ ∴ $d\lambda$ depends

But $d\lambda$ depends on u^α on curve to curve

⇒ Then we transport A^α to cross curve at $\lambda = 1+d\lambda$ to get $A^{\alpha'}(Q)$. Now $A^\alpha(Q)$ will be diff. than $A^{\alpha'}(Q)$. ∴ $\mathcal{L}_u A^\alpha(P) \equiv \frac{A^\alpha(Q) - A^{\alpha'}(Q)}{d\lambda}$

$= u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha$

If $A^\alpha(Q) = A^{\alpha'}(Q)$ then it is lie Transf.

⇒ In similar way we can find

$\mathcal{L}_u A_\alpha$, $\mathcal{L}_u A^{\alpha\beta}$ ⟹ From this we can see $\mathcal{L}$ follows Product Rule! (28)

⇒ How to make Integral Curves.

$x^\alpha(\lambda)$: $\frac{dx^\alpha}{d\lambda} = u^\alpha$; u^α is given, we have to find out dx^α.

This is 1st order DE ∴ Needs Initial condition so to form Integral Curve.

(2) ~~By Infinitesmal Coordinate Transfat.~~

(2) By Displacement vector Transportation.

⇒ $u_i \xi^i$ = const. all along geodesic

ξ^i = killing vector field

AS ξ^i are assumed to be ~~along~~ all our spacetime.

(64) Killing vectors can be used to find constants associated with the motion along a geodesic.

Let u^i = Tangent to geodesic $u^i(\lambda)$

$\boxed{\text{Assuming geodesic is affinely parameterize}}$

∴ $u^i \nabla_i u^a = 0$

otherwise $u^i \nabla_i u^a = k u^a$

ξ_i = killing vector

$\boxed{\text{\& spacetime is symmetric.}}$

$$\frac{d}{d\lambda}(u^i \xi_i) = \partial_\beta (u^i \xi_i) u^\beta$$

$$= \nabla_\beta (u^i \xi_i) u^\beta$$

$$= (u^\beta \nabla_\beta u^i)\xi_i + u^i u^\beta \nabla_\beta \xi_i$$

$$= 0$$

∴ $u^i \xi_i$ is constant along geodesics

In similar vein, P_a is conserved if $\nabla_a T^{ab} = 0$

$\nabla_a P^a = 0$

see (58)

$\rightarrow$ $\nabla_a T^{ab} = 0$ if EOM are satisfied

$\rightarrow$ if EOM satisfied $\Rightarrow$ $\nabla_a T^{ab} = 0$

(65) Another consequence of this is:

Doubt: ~~Thi~~ if the geodesic is affinely parameterized then if the geodesic is timelike at one pt. then all along the curve it remains timelike.

Proof

$$\frac{d(u_i u^i)}{d\lambda} = u^\beta \partial_\beta (u_i u^i)$$

$$= u^\beta \nabla_\beta (u_i u^i)$$

$u_i = \frac{dx_i}{d\lambda}$

$$= u^\beta u_i (\nabla_\beta u^i) + u^\beta u^i (\nabla_\beta u_i)$$

λ: is affinely parameterized.

$$= 2 u^\beta u_i \nabla_\beta u^i$$

$$= 0$$

$$u_i u^i = \text{const}$$

$$dx_i dx^i = \text{const}$$

Isn't $u_i u^i$ for affinely parameterization any ways zero? Eg. $\frac{dx^i}{d\lambda}\frac{dx_i}{d\lambda} = \frac{d\tau^2}{d\lambda^2} = \frac{1}{a^2}$

$\therefore \frac{d}{d\lambda}(1/a^2) = 0$?

$\rightarrow$ Another way

for Geod. Eqn.

$$\frac{d}{d\lambda}(u^i \xi_i) = \frac{D(u^i \xi_i)}{d\lambda} = \xi_i \cancel{\frac{Du^i}{d\lambda}} + u^i \frac{D\xi_i}{d\lambda}$$

$$= \xi_i u^\beta \nabla_\beta u^i + u^i u^\beta \nabla_\beta \xi_i$$

$$= 0$$

(66) Static, spherically symmetric spacetime. (28)

$$ds^2 = -f(r)dt^2 + f^{-1}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Metric g_{ij} ind. of t, ϕ.

$\therefore$ static & spherical sym.

$$f = 1 - \frac{2m}{r}$$

$\therefore$ 2 killing vectors

$$\xi^\alpha_t = \frac{dx^\alpha}{dt} = \overset{t\ \ r\ \ \theta\ \ \phi}{(1, 0, 0, 0)}$$

$$\xi^\alpha_\phi = \frac{dx^\alpha}{d\phi} = (0, 0, 0, 1)$$

$$\mathcal{L}_{\xi(t)} g_{ij} = \xi^{(\alpha)}_t \partial_\alpha g_{ij} + g_{\alpha j}\partial_i \xi^\alpha_{(t)} + g_{i\alpha}\partial_j \xi^\alpha_{(t)}$$

$$\overset{*}{=} \partial_t g_{ij} \overset{*}{=} 0$$

But $\mathcal{L}_{\xi(t)} g_{ij}$ is tensor $\therefore$

$$\mathcal{L}_{\xi(t)} g_{ij} = 0$$

Similarly $\mathcal{L}_{\xi_\phi} g_{ij} = 0$

By (64)

$$\left.\begin{aligned} \tilde{E} &= u_\alpha \xi^\alpha_{(t)} \\ \tilde{L} &= u_\alpha \xi^\alpha_{(\phi)} \end{aligned}\right\} \text{constants along geodesic}$$

$\dfrac{\text{Energy}}{\text{Mass}}$

$\dfrac{\text{Ang. mom}}{\text{mass}}$.

Axial Symmetry

(67) ξ_ϕ tells we can Rotate our spacetime in particular direction & further there are 2 more killing vectors which tell we can Rotate in 2 other directions as there is Spherical symmetry.

(69) $A^\alpha = \eta^{\alpha\beta} A_\beta$ (flat spacetime)

$A^\alpha = g^{\alpha\beta} A_\beta$ (GR)

$A^\alpha = \delta^{\alpha\beta} A_\beta$ (Euclidean)

$A^0 = A_0$, $A^1 = -A_1$ (STR)

$A^1 = A_1$, $A^2 = A_2$ (Euclidean)

(70) <u>No. of Independent Components of Tensor</u>

<u>Symmetric Tensor</u> : N Dim s index.

<u>As the order of Indices in Symmetric Tensor doesn't matter</u>

$T^{ijkl} \equiv T^{1233} = T^{1323} = T^{3123} = T^{3132}$

∴ The problem reduces to how many different ways there are to put s undistinguishable objects into d boxes.

There are d boxes an s balls, no. of balls B/w 2 bars determine no. of balls in box.

Eg. $\perp$ O | | O O O | |

$d + s = 5 + 4$

T^{0103}

0 Ball in 1 box

1 Ball in 2nd box

0 Ball in 3rd Box

No. of Ind. Components $= {}^{N+S-1}C_S$

(-1) Because last Bar is fixed.

Example

T^{ij} 4D & 28

0 | | | 0 |

T^{14}

T^{41} will also have same figure

$\therefore$ No. of Ind. comp. $T^{ij} = {}^{6-1}C_2 = {}^5C_2$

Express these from 1, 2, 3, 4 if we have to go to usual 0, 1, 2, 3 just sub -1 from above.

Antisymmetric Tensor

2 Ind. same then $T^{ii} = 0$

$\therefore$ No. of Ind. comp. $= {}^NC_s$

If $s = N$ then only one Ind. component.

When $s > N$ then Indep. comp. vanish.

(71) $s = N$ Ant. sym. $\equiv \epsilon$

Only one Ind. comp.

$\therefore$ Only one Relⁿ has to be defined all others relation can be found out.

Let $\epsilon_{123} = 1$

$\epsilon_{0123} = 1$

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(72) 3-vector can be found by ϵ_{ijk} (22)

$$C_i = \epsilon_{ijk} A^j B^k$$
$$= \frac{\epsilon_{ijk}}{2} (A^j B^k - A^k B^j)$$
$$\equiv \frac{\epsilon_{ijk}}{2} C^{jk} = \frac{(*C)_i}{2}$$

$(C^{jk})^* =$ Dual of C^{jk} $= A_i =$ Dual of C^{jk} is 3-vector

(73) Dual of 3-vector can be obtained as 3rd Rank Tensor

$$B_{ijk} = \epsilon_{ijkl} A^l = (*A)_{ijk}$$

$$B_{ij} = \epsilon_{ijkl} A^{kl} = (*A)_{ij}$$

(74) ~~Product~~

3-vector can be found by ϵ_{ijk} 3-vector can't be 4-vector or higher dim. vector can't be found.

$\rightarrow$ It is just coincidence.

$$C_i = \epsilon_{ijk} A^j B^k$$
$$= \frac{\epsilon_{ijk}}{2} C^{jk} \rightarrow 2^{nd} \text{ Rank Ant.}$$

$\therefore$ ${}^3C_2 = 3$

Same No. of components of vector.

In 4D

$$C_i = \epsilon_{ijk} A^j B^k$$
$$= \frac{\epsilon_{ijk}}{2} C^{jk} \rightarrow {}^4C_2 = 6$$

Not Equally

$\therefore$ Can't express

(75) Product of ϵ.

in 3D

$$\epsilon_{ijk}\,\epsilon^{prs} = \begin{vmatrix} \delta_i^p & \delta_i^r & \delta_i^s \\ \delta_j^p & \delta_j^r & \delta_j^s \\ \delta_k^p & \delta_k^r & \delta_k^s \end{vmatrix}$$

$$\epsilon_{ijk}\,\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_k^m \delta_j^n$$

$$\epsilon_{ijk}\,\epsilon^{ijm} = 2\delta_k^m$$

$$\epsilon_{ijk}\,\epsilon^{ijk} = 6$$

in 4D

$$\epsilon_{ijkl}\,\epsilon^{prst} = \begin{vmatrix} \delta_i^p & \delta_i^r & \delta_i^s & \delta_i^t \\ \delta_j^p & \delta_j^r & \delta_j^s & \delta_j^t \\ \delta_k^p & \delta_k^r & \delta_k^s & \delta_k^t \\ \delta_l^p & \delta_l^r & \delta_l^s & \delta_l^t \end{vmatrix}$$

$$\epsilon^{iklm}\,\epsilon_{prlm} = -2(\delta_p^i \delta_r^k - \delta_r^i \delta_p^k)$$

$$\epsilon^{iklm}\,\epsilon_{pklm} = -6\delta_p^i \,; \quad \epsilon^{iklm}\,\epsilon_{iklm} = -24$$

(76) Determinant can also be defined by ϵ.

$$\boxed{\epsilon^{prst} A_{ip} A_{kr} A_{ls} A_{mt} = +A\,\epsilon_{iklm}}$$

$$\epsilon^{iklm}\,\epsilon^{prst} A_{ip} A_{kr} A_{ls} A_{mt} = 24A.$$

$$A = |A_{ij}|$$

(77) **Th.** $|A| = |A^T|$

i.e. $|A_{ij}| = |A_{ji}|$

Proof: $\epsilon^{ijkl} \epsilon^{prst} A_{ip} A_{jr} A_{ks} A_{lt} = 24A$

$$\epsilon^{prst} \epsilon^{ijkl} A_{pi} A_{rj} A_{sk} A_{tl} = 24A$$

But LHS is $24A^T$ where $A^T = |A_{ji}|$

$$\therefore 24A^T = 24A$$

$$|A^T| = |A|$$

(78) Volume & Surface Integrals in flat Spacetime

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

J is $(m \times n)$ matrix

$$J_{ij} = \frac{\partial f_i}{\partial x^j}$$

when $\left.\begin{matrix} m = 1 \\ n = n \end{matrix}\right\} \Rightarrow J = (\vec{\nabla} f)$

$$d^4x = J(x') \, d^4x' = \left| \frac{\partial x}{\partial x'} \right| d^4x'$$

(79) $dx^i = L^i_{i'} \, dx^{i'}$

$$d^4x = \left| \frac{\partial x}{\partial x^{i'}} \right| d^4x^{i'} = |L^i_{i'}| \, d^4x^{i'}$$

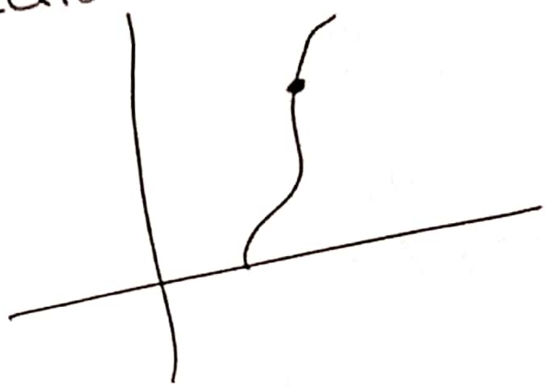
$\det L = 1 \quad \therefore d^4x = d^4x'$

(80) 3-dim. Surface in a 4-D space described in paramateric form by 4 functions

$x^i = x^i(a,b,c)$ with 3 parameters.

In the same way,

A curve (1D) in 2D can be described in parametric form by 2 functions using one parameter. $x(\tau), y(\tau)$



An inf. vol in 3D subspace is

(81) $$d^3\sigma_i = \frac{\epsilon_{ijkl}}{3!} J_{3D}\, da\, db\, dc$$

$$J_{3D} = \left[\frac{\partial(x^j, x^k, x^l)}{\partial(x^a, x^b, x^c)}\right] = \begin{vmatrix} \partial_a x^j & \partial_b x^j & \partial_c x^j \\ \partial_a x^k & \partial_b x^k & \partial_c x^k \\ \partial_a x^l & \partial_b x^l & \partial_c x^l \end{vmatrix}$$

(82) if the normal is timelike then surface is spacelike

$\therefore x_0 = \text{const.}$ is spacelike hypersurface.

let a, b, c parameters be x^1, x^2, x^3

$$d^3\sigma_i = \frac{\epsilon_{ijkl}}{3!}\left[\frac{\partial(x^j, x^k, x^l)}{\partial(x^1, x^2, x^3)}\right] d^3x$$

for each value of i, there are 3! arrangements of j, k, l which are not equal = i

$\therefore$ Ignore 3! in denominator

$$d^3\sigma_\alpha = \frac{\epsilon_{\alpha jkl}}{3!}\left[\frac{\partial(x^j, x^k, x^l)}{\partial(x^1, x^2, x^3)}\right] d^3x$$

α = Spatial Indices

$$J_{3D} = \begin{vmatrix} \partial_1 x^0 & \partial_2 x^0 & \partial_3 x^0 \\ \partial_1 x^k & \partial_2 x^k & \partial_3 x^k \\ \partial_1 x^l & \partial_2 x^l & \partial_3 x^l \end{vmatrix}$$

one index has to be 0.

$$= \begin{vmatrix} 0 & 0 & 0 \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{vmatrix} = 0$$

$\therefore\ d^3\sigma_\alpha = 0$

$\therefore\ d^3\sigma_0 = \epsilon_{0123}\, J_{3D}\, d^3x$

$= 1.1.d^3x$

$d^3\sigma_0 = d^3x$

(83) Similar for 2 Dim HyperSurface in 4D

$$d^2\sigma_{ij} = \frac{\epsilon_{ijkl}}{2!}\left[\frac{\partial(x^k, x^l)}{\partial(x^a, x^b)}\right] da\, db$$

(84) Divergence in 3D $\vec{\nabla}\cdot\vec{A}$

Divergence in 4D flat $\partial_i A^i$

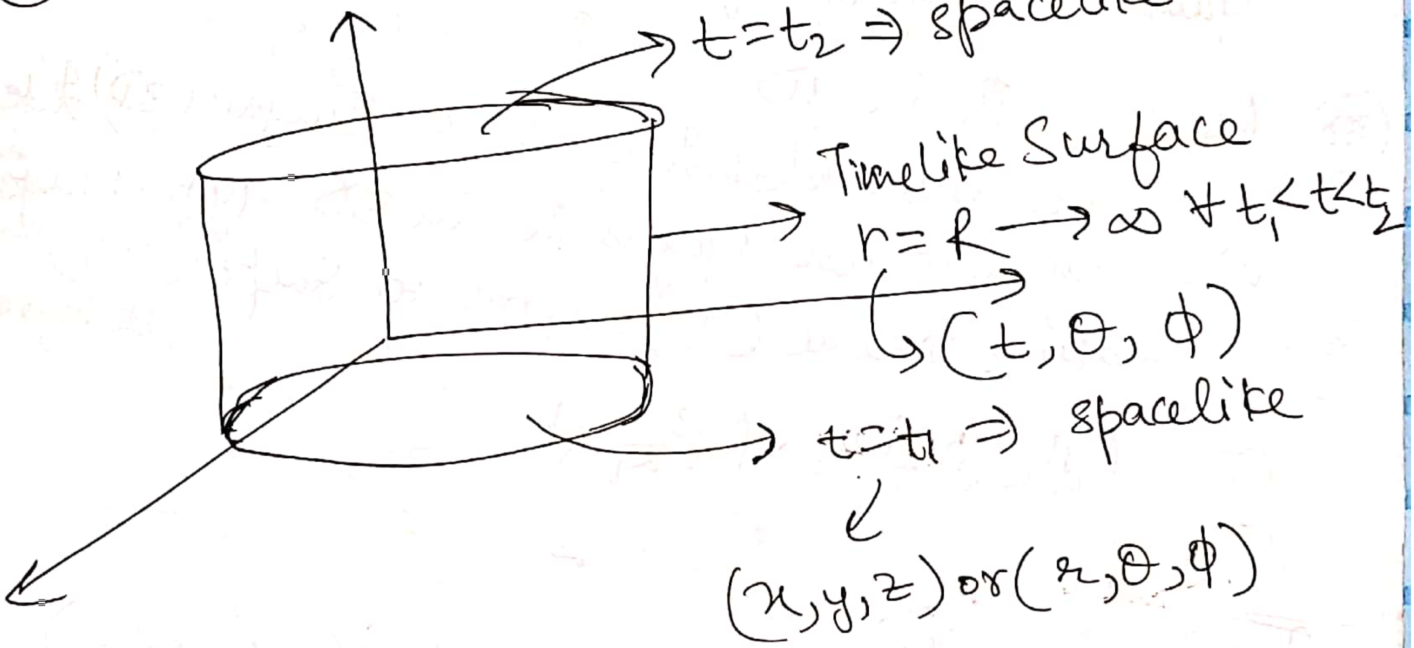
Gauss Th. in 3D $\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \int \vec{A}\cdot\vec{n}\, d^2x$

Gauss Th. in 4D $\int_V \partial_i A^i\, d^4x = \int_{\partial V} A^i\, d^3\sigma_i$

$$\int_V \partial_i A^i \, d^4x = \int_{\partial V} A^i \, d^3\sigma_i$$

LHS is 4D volume Integral & RHS is 3D surface Integral.

(85) Boundaries of $V = \partial V$



In RHS $\int_{\partial V} A^i \, d^3\sigma_i$

Integral has to be taken over surfaces
Assuming $A^i \to 0$ as $R \to \infty$ $\therefore$ $\int_R A^\alpha d^3\sigma_\alpha \to 0$

$\uparrow$ Spatial

$\therefore$ In RHS left term is

$$\int_{t=t_1}^{t=t_2} A^0 \, d^3\sigma_0 = \int_{t=t_1}^{t=t_2} A^0 \, d^3\vec{x} = \int_{t=t_2} A^0 d^3\vec{x} - \int_{t=t_1} A^0 d^3\vec{x}$$

If $\partial_i A^i = 0$
then A^0 is Conserved

86 The same Conservation be obtained by $\partial_i A^i = \partial_0 A + \vec{\nabla}\vec{A}^3 = 0$ [Continuity Eqn] (38)

87 Gauss Theorem has nothing to do with transformation property of A^i

Gauss Th. holds for any 4 fn (f^0, f^1, f^2, f^3)

88 As Gauss Th. in 4D makes Volume (4D) Integral to Surface (3D) Integral

We can also use Gauss Th. in 3D Hyp. Surf. making Volume (3D) Integral to Surface (2D) Integral

(1) $\int_V d^4x\, \partial_i A^i = \int_{\partial V} d^3\sigma_i\, A^i$

Doubt? (2) from 83

Inf. Volume in 2D $= d^2\sigma_{ij}$ (A^{ik}: Antisym)

$$\int_V d^3\sigma_i (\partial_k A^{ik}) = \int_{\partial V} d^2\sigma_{ij} \frac{A^{ij}}{2}$$

Proof:

$$\frac{1}{2}\int_{\partial V} A^{ik}\, d^2\sigma_{ik} = \frac{1}{2}\int_V \left(d^3\sigma_i \partial_k A^{ik} - d^3\sigma_k \partial_i A^{ik}\right)$$

$$d^2\sigma_{ik} \rightarrow d\sigma_i \partial_k - d\sigma_k \partial_i$$

$$= \int_V d^3\sigma_i (\partial_k A^{ik})$$

(39)

(89) If J^i is conserved $\partial_i J^i = 0$

$$\therefore \int_{\partial V} d^2\sigma_i J^i = \text{const.}$$

We can always find A^{ij} (Antisym) $J^i = \partial_k A^{ik}$

~~s.t.~~

$$\therefore \int d^3\sigma_i J^i = \int d\sigma_i (\partial_k A^{ik}) = \frac{1}{2}\int A^{ik} d\sigma_{ik}$$

(90) $$d^4x = J(x')\, d^4x' = \left[\frac{\partial x}{\partial x'}\right] d^4x'$$

$$\cancel{g_{i'j'} = \partial_{i'} x^i \partial_{j'} x^j g_{ij}}$$

$$\cancel{g' = J^2 g}$$

Now we know $x = f(x')$ from $g_{i'j'}$ we can find g_{ij}

we know this

$$\therefore g_{ij} = \partial_i x^{i'} \partial_j x^{j'} g_{i'j'}$$

$$g = J^2(x)\, g'$$

We know $J(x)\, J(x') = I$

$$\therefore J(x) = J^{-1}(x')$$

$$\sqrt{-g} = \frac{\sqrt{-g'}}{J(x')}$$

$$\therefore d^4x = \frac{\sqrt{-g'}}{\sqrt{-g}} d^4x'$$

$$\boxed{\sqrt{-g}\, d^4x = \sqrt{-g'}\, d^4x'}$$

(91) In comparison to (90)

$$d^4x' = J(x)\, d^4x = \left[\frac{\partial x'}{\partial x}\right] d^4x$$

Now we know $x' = f(x)$

$\therefore$ we know g_{ij} & we can find $g_{i'j'}$

$$g_{i'j'} = \partial_{i'} x^i \, \partial_{j'} x^j \, g_{ij}$$

$$g' = J^2(x')\, g$$

$$\sqrt{-g'} = \frac{\sqrt{-g}}{J(x)}$$

$$\therefore \; d^4x' = \frac{\sqrt{-g}}{\sqrt{-g'}}\, d^4x$$

$$\boxed{\sqrt{-g'}\, d^4x' = \sqrt{-g}\, d^4x}$$

(92) As in SR

$$d^3\sigma_i = \frac{\epsilon_{ijkl}}{3!}\, J_{3D}\, da\, db\, dc$$

$$= \frac{\epsilon_{ijkl}}{3!}\left[\frac{\partial(x^j, x^k, x^l)}{\partial(x^a, x^b, x^c)}\right] da\, db\, dc$$

$$d^3\sigma_{i'} = \frac{\epsilon_{i'j'k'l'}}{3!}\left[\frac{\partial(x^{j'}, x^{k'}, x^{l'})}{\partial(x^{a'}, x^{b'}, x^{c'})}\right] da'\, db'\, dc'$$

But in SR

$d^4x = d^4x'$ as $J = 1$

$\therefore d^3\sigma_i = d^3\sigma_{i'}$

$da\,db\,dc = da'\,db'\,dc'$

$J = 1$

$\therefore \epsilon_{ijkl} = \epsilon_{i'j'k'l'}$

(93) But under general transf. components change as $J \neq 1$ $\therefore$ under general Transf. ϵ is not a Tensor

Now let $\epsilon_{ijkl} = [ijkl]$

$$\epsilon^g_{abcd} = +\sqrt{-g}\,[abcd] \qquad J(x')$$

general cov. Tensor $\equiv$ Levi Civita Tensor

Proof: $[abcd]\,\partial_{a'}x^a\,\partial_{b'}x^b\,\partial_{c'}x^c\,\partial_{d'}x^d = +\left[\partial_{a'}x^a\right][a'b'c'd']$

from (76) As RHS is comp. Antisy.

$\therefore$ LHS is completely AS.

$$J = \begin{vmatrix} \partial_{a'}x^a & \partial_{b'}x^a & \partial_{c'}x^a & \partial_{d'}x^a \\ & & & \\ & & & \end{vmatrix}$$

let $[a'b'c'd] = [0\,1\,2\,3] = 1$

$\therefore J(x') = +[abcd]\,\partial_{0'}x^a\,\partial_{1'}x^b\,\partial_{2'}x^c\,\partial_{3'}x^d$

Now we know $x^a = f(x^{a'})$

$\therefore$ we know $g_{i'j'}$ we can get g_{ij}

$$g_{ij} = \partial_i x^{i'}\,\partial_j x^{j'}\,g_{i'j'}$$

$$\sqrt{-g} = J(x)\sqrt{-g'} = \frac{\sqrt{-g'}}{J(x')}$$

$\therefore \sqrt{-g}\,[abcd]\,\partial_{a'}x^a\,\partial_{b'}x^b\,\partial_{c'}x^c\,\partial_{d'}x^d = -\sqrt{-g'}\,[a'b'c'd']$

$\therefore \epsilon^g_{a'b'c'd'} = \epsilon^g_{abcd}\,\partial_{a'}x^a\,\partial_{b'}x^b\,\partial_{c'}x^c\,\partial_{d'}x^d$

$\therefore \epsilon^g_{abcd}$ is a Tensor

(94) $\epsilon_g^{ijkl} = -\dfrac{[ijkl]}{\sqrt{-g}}$

(95) As ϵ_{ijke} is Tensor (43)

$$\therefore \epsilon_{ijke} = g_{i\alpha} g_{j\beta} g_{kr} g_{e\delta} \epsilon^{\alpha\beta r\delta}$$

Which can be checked as

$$\epsilon_{0123} = g_{0\alpha} g_{1\beta} g_{2r} g_{3\delta} \frac{[\alpha\beta r\delta]}{\sqrt{-g}}$$

from (76)

$$= -\frac{g}{\sqrt{-g}} = \frac{+ig}{\sqrt{g}} = \sqrt{-g}$$

$$\therefore \epsilon_{0123} = \sqrt{-g}$$

which is correct from def. $\epsilon_{ijke} = \sqrt{-g}\,[ijke]$

$$\therefore \epsilon_{ijke} = g_{i\alpha} g_{j\beta} g_{k\gamma} g_{e\delta} \epsilon^{\alpha\beta\gamma\delta}.$$

(96) Imp

$$\nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu = -R^\mu{}_{\nu\alpha\beta} A^\nu$$

$$A^\mu{}_{;ij} - A^\mu{}_{;ji} = -R^\mu{}_{\ell ij} A^\ell$$

$$\nabla_i \nabla_j A_\mu - \nabla_j \nabla_i A_\mu = R^\alpha{}_{\mu ij} A_\alpha$$

$$\boxed{\nabla_i \nabla_j T^\alpha_\beta - \nabla_j \nabla_i T^\alpha_\beta = -R^\alpha{}_{rij} T^r_\beta + R^r{}_{\beta ij} T^\alpha_r}$$

(97) <u>Properties of R</u>

$R_{abcd} = -R_{abdc}$ (can be seen directly)

All other prop. of R_{abcd} can be found in LIf.

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = R_{cdab}$$

$$R_{a[bcd]} = 0 \quad (\text{Anti Sym})$$

$$\nabla_{[i} R^a{}_{b|cd]} = 0 \quad (\text{Bianchi Idities})$$

Scanned with CamScanner

98 $R_{\alpha\beta} = R^{\mu}_{\ \alpha\mu\beta}$

$R = R_{\alpha\beta}\, g^{\alpha\beta}$

99 $G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta}$ (Einstein Tensor)

from Bianchi Identities we can get

contracted Bianchi Identity $\nabla_\alpha G^{\alpha\beta} = 0$

→ Sym. (10 Ind)

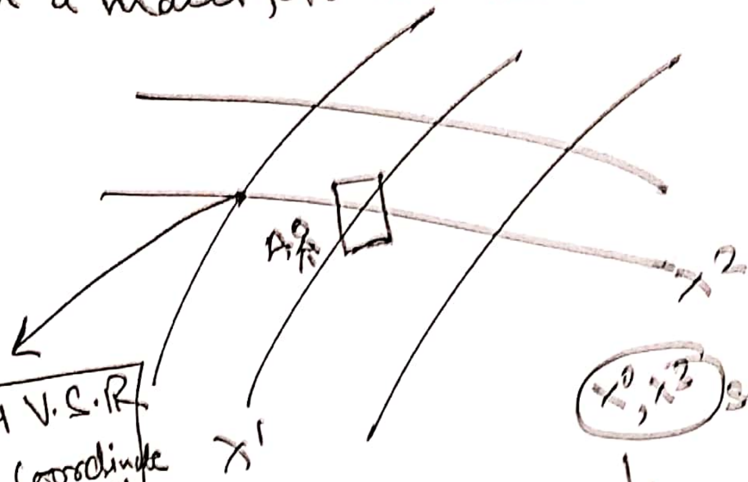
100 Einstein field Eqⁿ $G^{\alpha\beta} = 8\pi\, T^{\alpha\beta}$

↑ Spacetime ↑ Matter

Tetrad formalism

① Spacetime = Manifold + Metric

On a manifold we can use coordinate system.



A has a small lab. Small lab means geodesic not crossing each other i.e. congruence.

T_p 4 V.S.R
∂_μ = Coordinate Basis

T_p^*
dx^μ = Coord Basis

(x^0, x^3) suppressed

Bird's view coordinate

at each point

$\therefore$ at A pt. also
∂_μ = coordinate Basis

But
These coord. Basis are not necessarily orthonormal.

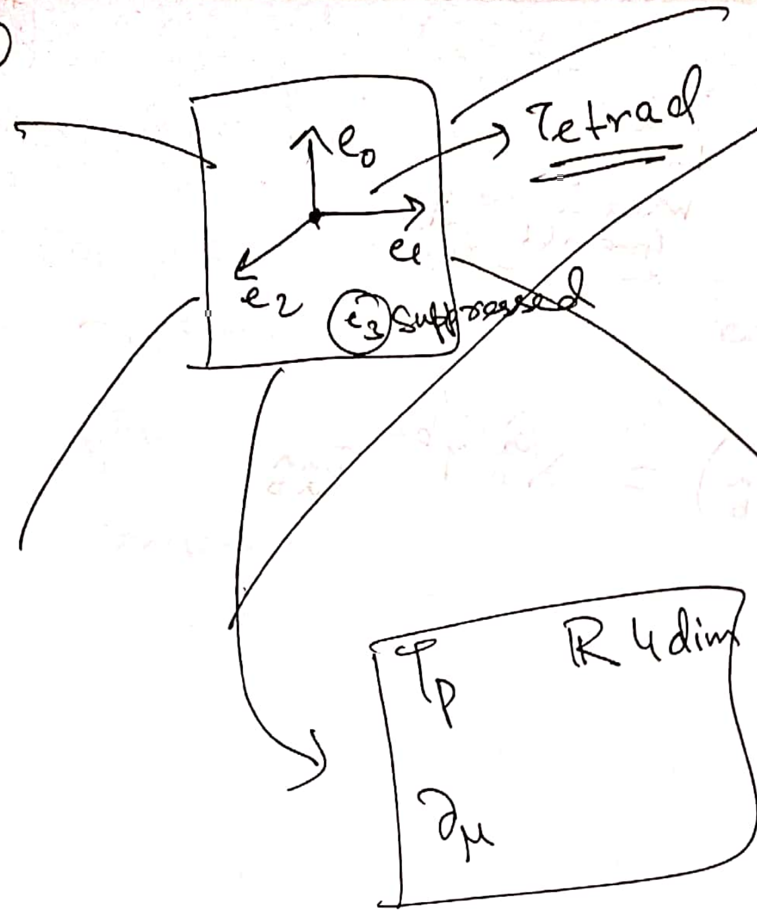
$$g(\partial_\mu, \partial_\nu) \neq \eta_{\mu\nu}$$

If $g(\partial_\mu, \partial_\nu) = \eta_{\mu\nu} \Rightarrow$ Spacetime is flat

② $g_{\mu\nu} = g(\partial_\mu, \partial_\nu) \equiv$ Components of metric Tensor

③ A wouldn't use global Basis as they are not orthogonal.
He would prefer to use orthon. Basis of his own.

④ ⑬



A coordinates Basis in terms of coord. Basis of Bird view

⑤ $e_{\hat{0}} \in T_p$

$\therefore e_{\hat{0}} = \boxed{}\, \partial_\mu$

$= e^{\mu}_{\hat{0}}\, \partial_\mu$

$e_{\hat{1}} = e^{\mu}_{\hat{1}}\, \partial_\mu$

$e_{\hat{2}} = e^{\mu}_{\hat{2}}\, \partial_\mu$

$e_{\hat{3}} = e^{\mu}_{\hat{3}}\, \partial_\mu$

$\boxed{e_{\hat{a}} = e^{\mu}_{\hat{a}}\, \partial_\mu}$

Linear Transfn.

⑥ Tetrads vectors are with a hat.

⑦ $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$

As $g(0,0)$ can take any vectors

$g(e_{\hat{a}}, e_{\hat{b}}) = g_{\hat{a}\hat{b}}$

But construction they all Orthogonal

$\therefore g(e_{\hat{a}}, e_{\hat{b}}) = \eta_{\hat{a}\hat{b}}$

But $e_{\hat{0}}, e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}}$ are also Basis of T_p along with ∂_μ.

$\therefore$ let $X \in T_p$

$X = x^\mu \partial_\mu = X^{\hat{a}} e_{\hat{a}}$

⑧ $g(e_{\hat{a}}, e_{\hat{b}}) = g(e^{\mu}_{\hat{a}}\partial_\mu, e^{\nu}_{\hat{b}}\partial_\nu)$

$= e^{\mu}_{\hat{a}} e^{\nu}_{\hat{b}}\, g(\partial_\mu, \partial_\nu)$

$\eta_{\hat{a}\hat{b}} = e^{\mu}_{\hat{a}} e^{\nu}_{\hat{b}}\, g_{\mu\nu}$

(9) $\underbrace{g_{\mu\nu}}_{\substack{\text{metric}\\ \text{component}\\ \text{from Bird's eye view}}} e^{\mu}_{\hat{a}} e^{\nu}_{\hat{b}} = \underbrace{\eta_{\hat{a}\hat{b}}}_{\substack{\text{metric comp.}\\ \text{for alice (A)}}}$

(10) $g(x^{\hat{a}} e_{\hat{a}}, y^{\hat{b}} e_{\hat{b}}) = x^{\hat{a}} y^{\hat{b}} \eta_{\hat{a}\hat{b}}$

Ch-2

Geodesic Congruences

2.1 → 9.2 wald
Ch-12 Visser

① For material particles

$A_m = -m \int d\tau = -m \int \sqrt{g_{ij}\, dx^i\, dx^j}$ 2.3, 2.4 → 9.2, Ap-B wald
6.1 Carter

$\delta_g A_m = \int \delta g^{ab} \frac{T_{ab}}{2} \sqrt{g}\, d^4x$

comparing

we get $\underbrace{T^{ab} = \rho u^a u^b}_{\text{fluid with density } \rho}$

② In the Rest frame of that fluid LIf.

$T^{ab} u_b = \rho u^a = (\rho u^0, \rho u^\alpha)$ ——— (1)

$\boxed{\text{As this is Tensorial } \therefore \text{ valid in any frame} \therefore T^{ab}u_b \text{ is Mom. flux measured by observed with } u_b.}$

$= (\rho\gamma, \rho\gamma\vec{v})$

$\rho = T^{ab} u_a u_b$ ——— (2)

Energy Density   Momentum flux Density

In LIf Comoving $\rho = T^{00}$ = Energy Density

But $\rho = T^{ab} u_a u_b$ is Tensorial

$\therefore$ Energy Density $\equiv \rho$

③ In Relativity the above fluid is called Dust which has no p.

④ For ideal fluid, there is p & ρ for it.

$T^{ab} = (\rho + p) u^a u^b - p g^{ab}$

$\rho = T^{ab} \underline{u_a} \underline{u_b}$ (observer with 4 velocity $\underline{u_a}$ measures Energy Density ρ)

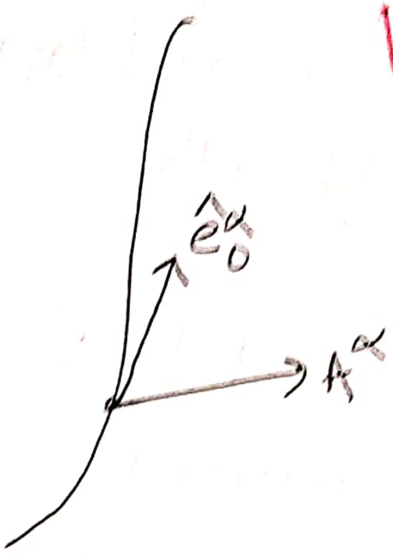
In Rest frame $\underline{u_a}$ (4-velocity of observer = fluid velocity) ↙ u_a

⑤ Assuming stress Tensor decompose into:

$$T^{\alpha\beta} = \rho\, \hat{e}_0^{\alpha} \hat{e}_0^{\beta} + p_1 \hat{e}_1^{\alpha} \hat{e}_1^{\beta} + p_2 \hat{e}_2^{\alpha} \hat{e}_2^{\beta} + p_3 \hat{e}_3^{\alpha} \hat{e}_3^{\beta}$$

↳ why is it Diagonal basis?
↓
Refer 9.2 Wald

For example:



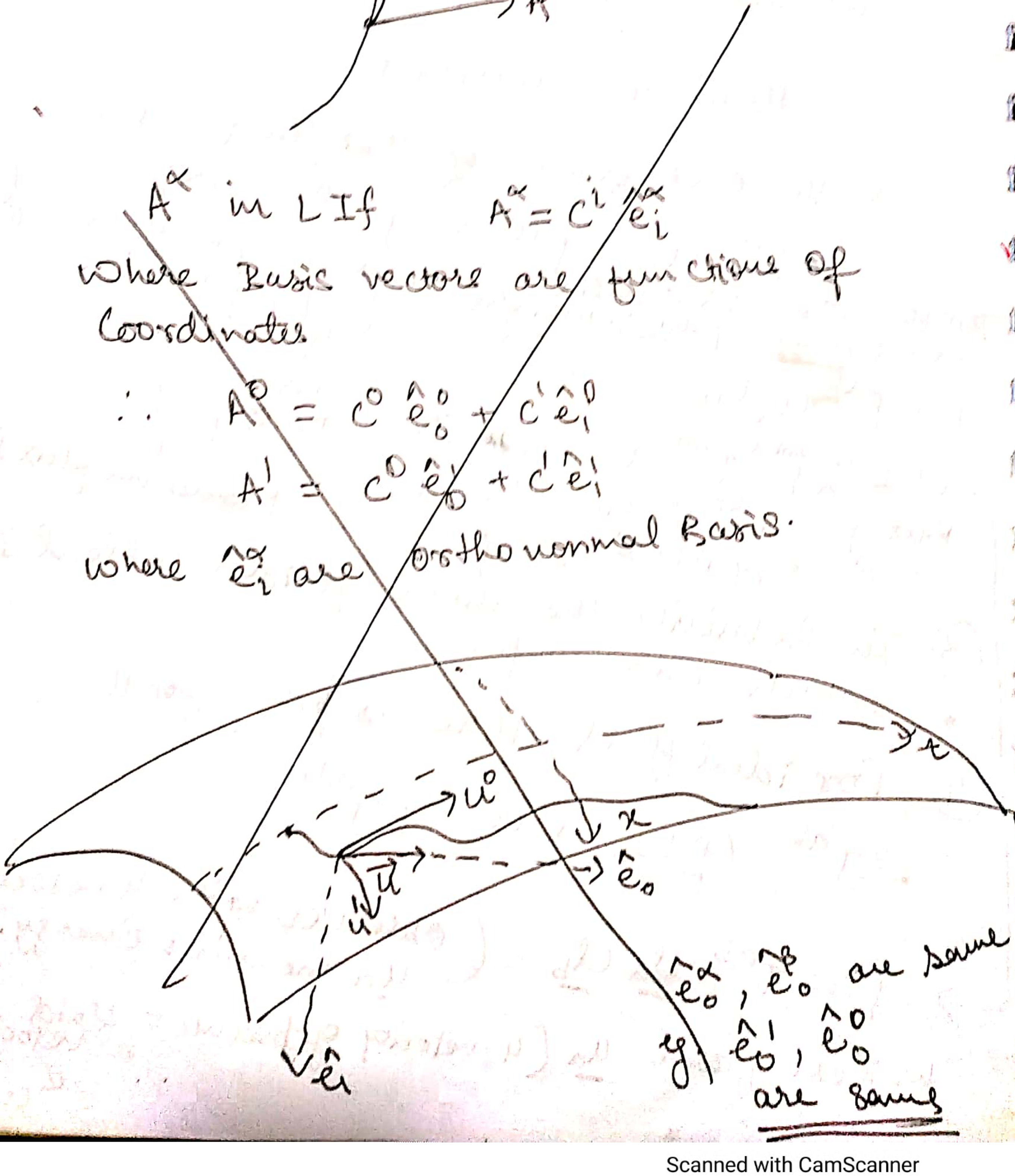
A^{α} in LIf $A^{\alpha} = c^{i} \hat{e}_i^{\alpha}$

where Basis vectors are functions of Coordinates

$\therefore\ A^{0} = c^{0} \hat{e}_0^{0} + c^{1} \hat{e}_1^{0}$

$A^{1} = c^{0} \hat{e}_0^{1} + c^{1} \hat{e}_1^{1}$

where $\hat{e}_i^{\alpha}$ are orthonormal Basis.



$\hat{e}_0^{\alpha}, \hat{e}_0^{\beta}$ are same

eg) $\hat{e}_0^{1}, \hat{e}_0^{0}$ are same

7 from 6 (57)

we get ρ, p_i are eigenvalues of $T_{\alpha\beta}$

Doubt

& $\hat{e}^{\alpha}_i$ are Normalized eigenvectors.

Inner product b/w orthonormal basis

8 ~~LIF~~ $g_{\alpha\beta} \to \eta_{ij}$

Doubt

$$g_{\alpha\beta}\,\hat{e}^{\alpha}_i\,\hat{e}^{\beta}_j = \eta_{ij}$$

Basis of the Spacetime

$$g^{r\alpha} g_{\alpha\beta}\,\hat{e}^{\alpha}_i\,\hat{e}^{\beta}_j = g^{r\alpha}\,\eta_{ij}$$

~~$e^{\alpha}_i e_{\alpha 1} = -1$~~
~~$e^{\alpha}_0 e_{\alpha 0} = 1$~~
~~$e^{\alpha}_i e_{\alpha 0} = 0$~~

here $a, b \in (1,2,3)$

e^{α}_i are the basis of H.S.

$$\eta^{li}\,\hat{e}^{\alpha}_i\,\hat{e}^{r}_j = \eta^{li}\,\eta_{ij}\,g^{r\alpha}$$

$$\eta^{li}\,\hat{e}^{\alpha}_i\,\hat{e}^{r}_j = \delta^{l}_j\,g^{r\alpha}$$

~~in LIF~~

$$\delta^{l}_j\,\eta^{li}\,\hat{e}^{\alpha}_i\,\hat{e}^{r}_j = g^{r\alpha}$$

$$\boxed{\eta^{li}\,\hat{e}^{\alpha}_i\,\hat{e}^{r}_l = g^{r\alpha}}$$

Completeness Relations

compare

$$h_{ab} = g_{\alpha\beta}\,e^{\alpha}_a\,e^{\beta}_b$$
$$h^{\alpha\beta} = h^{ab}\,e^{\alpha}_a\,e^{\beta}_b$$

9 Perfect fluid

$$p_1 = p_2 = p_3 = p$$

$$\therefore T^{\alpha\beta} = \rho\,\hat{e}^{\alpha}_0\,\hat{e}^{\beta}_0 + p\left(\hat{e}^{\alpha}_1\hat{e}^{\beta}_1 + \hat{e}^{\alpha}_2\hat{e}^{\beta}_2 + \hat{e}^{\alpha}_3\hat{e}^{\beta}_3\right)$$

$$= \rho\,\hat{e}^{\alpha}_0\,\hat{e}^{\beta}_0 + (-p)\left(g^{\alpha\beta} - \hat{e}^{\alpha}_0\,\hat{e}^{\beta}_0\right)$$

~~in LIF~~

$$T^{\alpha\beta} = (\rho + p)\,\hat{e}^{\alpha}_0\,\hat{e}^{\beta}_0 - p\,g^{\alpha\beta}$$

~~in LIF~~

~~$\hat{e}^{\alpha}_0$ is 4velocity of the fluid. as~~ ~~$u_0 = \hat{e}^{\alpha}_0$~~ ~~$u_i = 0$~~

$\hat{e}^{\alpha}_0$ is the 4vel. of fluid.

Compare with (4)

(10) Some energy conditions are formulated in terms of future directed timelike vector, V^a which represents 4 vel. of Arbitrary observer in the spacetime.

for future directed $\boxed{\frac{dt}{d\tau} = -\gamma}$ ~~-ve~~ +ve ∴ future Directed

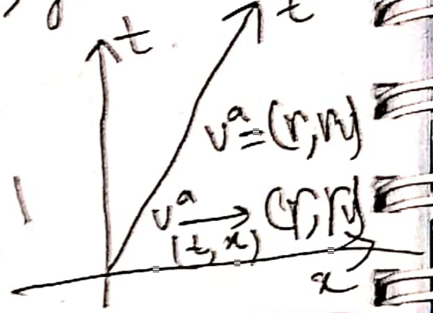
$$V^a = \frac{dx^a}{d\tau} \xrightarrow[\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3]{\text{in}} (\gamma, \gamma a, \gamma b, \gamma c)$$ ↳ just like

$V = V^i e_i$ or $V = V^i \hat{e}_i$

$$V^a = \gamma(\hat{e}_0^a + a\hat{e}_1^a + b\hat{e}_2^a + c\hat{e}_3^a)$$

$$\gamma = (1 - a^2 - b^2 - c^2)^{-1/2}$$

s.t. $a^2 + b^2 + c^2 < 1$



$V^a V_a = 1 \Rightarrow$ Normalized

(11) Future directed Null vector k^a

+ve

$$k^a = (\hat{e}_0^a + a'\hat{e}_1^a + b'\hat{e}_2^a + c'\hat{e}_3^a)$$ → why such?

$$a'^2 + b'^2 + c'^2 = 1$$

$k^a k_a = 0 \Rightarrow$ Null vector ∴ Normalization of Null vector is Arbitrary

(12) Weak Energy Condn

See (3) & (4)

For an observer with 4-velocity v^a Energy Density of fluid d's is

$$\rho = T_{\alpha\beta} v^\alpha v^\beta$$

Weak Energy Condn $\rho \geq 0$ (for any future Directed Timelike v^a)

What if Past Directed ?

(13) $v^\alpha = c^i \hat{e}_i^\alpha$

How to express v_α ?

(14) ~~$T_{\alpha\beta} v^\alpha v^\beta \geq 0$~~

$$T^{\alpha\beta} v_\alpha v_\beta \geq 0$$

$$(\rho e_0^\alpha e_0^\beta + p_1 e_1^\alpha e_1^\beta + p_2 e_2^\alpha e_2^\beta + p_3 e_3^\alpha e_3^\beta) g_{\alpha r} g_{\beta \Gamma} \gamma^2$$
$$(e_0^r + a e_1^r + b e_2^r + c e_3^r)(e_0^\Gamma + a e_1^\Gamma + b e_2^\Gamma + c e_3^\Gamma) \geq 0$$

$$\Rightarrow \rho e_0^r e_0^\alpha e_0^\beta e_0^\Gamma g_{\alpha r} g_{\beta\Gamma} = \rho \eta_{00} \eta_{00} = \rho$$

as $g_{\alpha r} e_i^\alpha e_j^\beta = \eta_{ij}$

$$\Rightarrow \rho a^2 e_0^\alpha e_0^\beta e_1^r e_1^\Gamma g_{\alpha r} g_\beta = \rho a^2 \eta_{01} \eta_{01} = 0$$

$$\therefore \rho \gamma^2 + p_1 a^2 \gamma^2 + p_2 b^2 \gamma^2 + p_3 c^2 \gamma^2 \geq 0$$

$$\gamma^2 (\rho + p_1 a^2 + p_2 b^2 + p_3 c^2) \geq 0$$

But $\gamma \geq 0$ $\boxed{\therefore \rho + p_1 a^2 + p_2 b^2 + p_3 c^2 \geq 0}$

$\gamma = (1 - a^2 - b^2 - c^2)^{-1/2}$ where $a^2 + b^2 + c^2 \leq 1$

(5) → let velocity of the object $= 0$ i.e. $a = b = c = 0$
↳ in tetrad frame

$\therefore \rho \geq 0$

→ let $b = c = 0$ where $a^2 < 1$

$$\rho + p_1 a^2 \geq 0$$

$$\therefore 0 \leq \rho + p_1 a^2 < \rho + p_1$$

$\therefore \rho + p_1 \geq 0$ & similar for p_2, p_3.

$\therefore$ Weak Energy Condition $\boxed{\begin{array}{l} \rho \geq 0 \\ \rho + p_i \geq 0 \quad \forall i \end{array}}$

(68)

(16) Null Energy Condition

Same as weak Energy But now Null vector, future directed is included. → why future directed?

k^α

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0$$

$$\left(\rho e_0^\alpha e_0^\beta + p_1 e_1^\alpha e_1^\beta + p_2 e_2^\alpha e_2^\beta + p_3 e_3^\alpha e_3^\beta\right)\left(g_{\alpha r} g_{\beta \Gamma}\right)$$

$$\left(e_0^r + a' e_1^r + b' e_2^r + c' e_3^r\right)\left(e_0^\Gamma + a' e_1^\Gamma + b' e_2^\Gamma + c' e_3^\Gamma\right) \geq 0$$

$$\rho e_0^\alpha e_0^\beta e_0^r e_0^\Gamma g_{\alpha r} g_{\beta\Gamma} = \rho$$

$$\therefore \quad \rho + p_1 a'^2 + p_2 b'^2 + p_3 c'^2 \geq 0$$

$$a'^2 + b'^2 + c'^2 = 1$$

Similar as previously

→ let $b' = c' = 0 \Rightarrow a' = 1$

$$\rho + p_1 \geq 0 \Rightarrow \boxed{\rho + p_i \geq 0 \quad \forall i}$$

(17) Strong Energy Condition

→ why?

$$R_{\alpha\beta} V^\alpha V^\beta \geq 0 \rightarrow \text{for future Directed Timelike vector}$$

(18) Einstein field Eqn

$$R_{ik} - \frac{g_{ik} R}{2} = 8\pi k T_{ik}$$

$$R^i_k - g^i_k \frac{R}{2} = 8\pi k T^i_k$$

$$g^k_i R^i_k - g^k_i g^i_k \frac{R}{2} = 8\pi k T^i_k g^k_i$$

$$R - (\delta^k_i \delta^i_k)\frac{R}{2} = 8\pi k T$$

$$\boxed{R = -8\pi k T}$$

$$\therefore \; R_{ik} + \frac{8\pi k g_{ik} T}{2} = 8\pi k T_{ik}$$

$$R_{ik} = 8\pi k \left(T_{ik} - \frac{g_{ik} T}{2} \right)$$

geometry sources.

(19) Strong Energy Conditions can be converted

$$\left(T_{\alpha\beta} - \frac{T g_{\alpha\beta}}{2}\right) v^\alpha v^\beta \geq 0 \equiv T_{\alpha\beta} v^\alpha v^\beta \geq -\frac{T}{2}$$

$$\equiv R_{ij} v^i v^j = 8\pi \left[T_{ij} v^i v^j + \frac{T}{2} \right]$$

Stresses should not become -ve

(20) ~~pe~~ Taking

$$\left(T^{\alpha\beta} - \frac{T g^{\alpha\beta}}{2}\right) v_\alpha v_\beta \geq 0$$

$$\left(\left(\rho e^\alpha_0 e^\beta_0 + p_1 e^\alpha_1 e^\beta_1 + p_2 e^\alpha_2 e^\beta_2 + p_3 e^\alpha_3 e^\beta_3\right) - \frac{T}{2} g^{\alpha\beta}\right) g_{\alpha r} g_{\beta \Gamma} \gamma^2$$

$$(e^r_0 + a e^r_1 + b e^r_2 + c e^r_3)(e^\Gamma_0 + a e^\Gamma_1 + b e^\Gamma_2 + c e^\Gamma_3) \geq 0$$

$$\rho + a^2 p_1 + b^2 p_2 + c^2 p_3 \geq \frac{T}{2} g_{r\Gamma} (e^r_0 + a e^r_1 + b e^r_2)(e^\Gamma_0 + a e^\Gamma_1 \ldots)$$

$$\frac{T}{2}(1 - a^2 - b^2 - c^2)$$

$$(\rho + a^2 p_1 + b^2 p_2 + c^2 p_3) \geq \frac{T}{2}\frac{1}{\gamma^2}$$

$$T = T^{\alpha\beta} g_{\alpha\beta} = g_{\alpha\beta}(\rho e_0^\alpha e_0^\beta + p_1 e_1^\alpha e_1^\beta + p_2 e_2^\alpha e_2^\beta + p_3 e_3^\alpha e_3^\beta)$$

$$= \rho - p_1 - p_2 - p_3$$

$$\therefore \boxed{\gamma^2(\rho + a^2 p_1 + b^2 p_2 + c^2 p_3) \geq \frac{(\rho - p_1 - p_2 - p_3)}{2}}$$

Strong cond^n.

(21)

$\Rightarrow$ let $a = b = c = 0 \Rightarrow \gamma = 1$

$$\boxed{\rho + p_1 + p_2 + p_3 \geq 0}$$

$\Rightarrow$ let $b = c = 0 \Rightarrow \gamma^2 = \frac{1}{1-a^2}$ where $a < 1$

$$\frac{(\rho + a^2 p_1)}{1 - a^2} \geq \frac{\rho - p_1 - p_2 - p_3}{2}$$

$$\cancel{2}\rho + \cancel{2}a^2 p_1 \geq \cancel{\rho} - p_1 - p_2 - p_3 - a^2\rho + \cancel{a^2 p_1} + a^2 p_2 + a^2 p_3$$

$$0 \geq \rho + p_1 + p_2 + p_3 \geq a^2(p_2 + p_3 - \rho - p_1)$$

$$a^2 < 1$$

$$a^2(p_2 + p_3 - \rho - p_1) \leq 0 \Rightarrow p_2 + p_3 \leq \rho + p_1$$

As $p_2 + p_3 + \rho + p_1 \geq 0 \Rightarrow 2(\rho + p_1) \geq 0$

$$\boxed{\Rightarrow \rho + p_i \geq 0 \ \forall i}$$

63.

(22) Strong condn doesn't imply weak condn
for null case Strong Energy Cond $\Rightarrow$ weak Null cond

(23) Dominant Energy Condition.

Matter should flow along timelike or null world line.

$\therefore$ Momentum flux Density as measured by observer with 4 velocity v^α should be timelike /Null.

$\Rightarrow T^{\alpha}_{\beta} v^\beta$ = Future Directed Time/Nulllike vector field

$\therefore T^{\alpha}_{\beta} v^\beta$ should not be spacelike.

$T^{\alpha\beta} v_\beta = X^\alpha$ $\qquad X^\alpha X_\alpha \geq 0$ (Time/Nulllike)

$T_{\alpha i} v^i = X_\alpha$

$\Rightarrow (T^{\alpha\beta} v_\beta)(T_{\alpha i} v^i) \geq 0$

$(\rho e_0^\alpha e_0^\beta - \cdots) g_{\beta r} \gamma (e_0^r + a e_1^r - \cdots) g_{\alpha a} g_{ib} (\rho e_0^a e_0^b - \cdots)$
$\gamma (e_0^i + a e_1^i - \cdots) \geq 0$

$$\boxed{\rho^2 - p_1^2 a^2 - p_2^2 b^2 - p_3^2 c^2 \geq 0}$$

(24) Let $a = b = c = 0$ $\qquad \gamma = 1$

$\therefore \rho^2 \geq 0$

$T^{\alpha\beta} v_\beta = T^{\alpha\beta} g_{\beta r} v^r = \rho e_0^\alpha - p_1 e_1^\alpha - p_2 e_2^\alpha - p_3 e_3^\alpha$
= Future Directed $\therefore \rho \geq 0$

let $b = c = 0$

$$\rho^2 \geq a^2 p_1^2 \qquad a^2 < 1$$

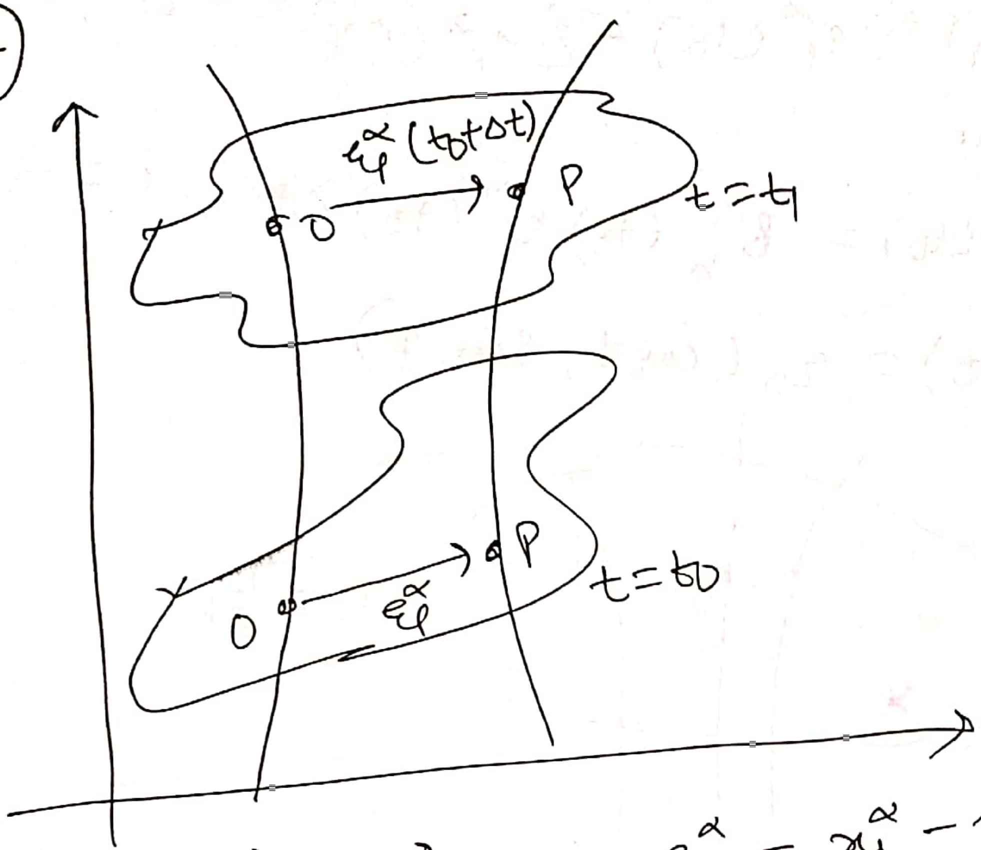
This Energy Condition Can be Interpreted as speed of energy flow of matter is < speed of light.

$$\rho \geq |p_i| \quad \forall \, i$$

2.5 Violations of the Energy Condition

Scanned with CamScanner

27



$$\vec{\xi_\varphi} = \vec{x_1} - \vec{x_0} \Rightarrow \xi_\varphi^\alpha = x_1^\alpha - x_0^\alpha$$

$$\frac{d\vec{\xi_\varphi}}{dt} = \vec{v}(x_1,t) - \vec{v}(x_0,t)$$

Equivalent to $\frac{\partial \xi_\varphi}{\partial t}$

Let P be near to O

$$\frac{d\xi_\varphi^\alpha}{dt} = v^\alpha(x_0+\xi_\varphi, t) - v^\alpha(x_0,t)$$
$$= v^\alpha(x_0) + \xi_\varphi^k \partial_k v^\alpha(x_0) + O(\xi_\varphi^2) - v^\alpha(x_0)$$

$$\frac{d\xi_\varphi^\alpha}{dt} \approx \xi_\varphi^k \partial_k v^\alpha(x_0) = \xi_\varphi^k B_k^\alpha$$

28 $f(x) = f(x_0) + (x-x_0) f'(x_0) + (x-x_0)^2 \frac{f''(x_0)}{2!}$

Imp

$$v^\alpha(x_0+\xi_\varphi) = v^\alpha(x_0) + (x_0+\xi_\varphi^k - x_0)\partial_k v^\alpha + O(\xi_\varphi^2)$$
$$= v^\alpha(x_0) + \xi_\varphi^k \partial_k v^\alpha$$

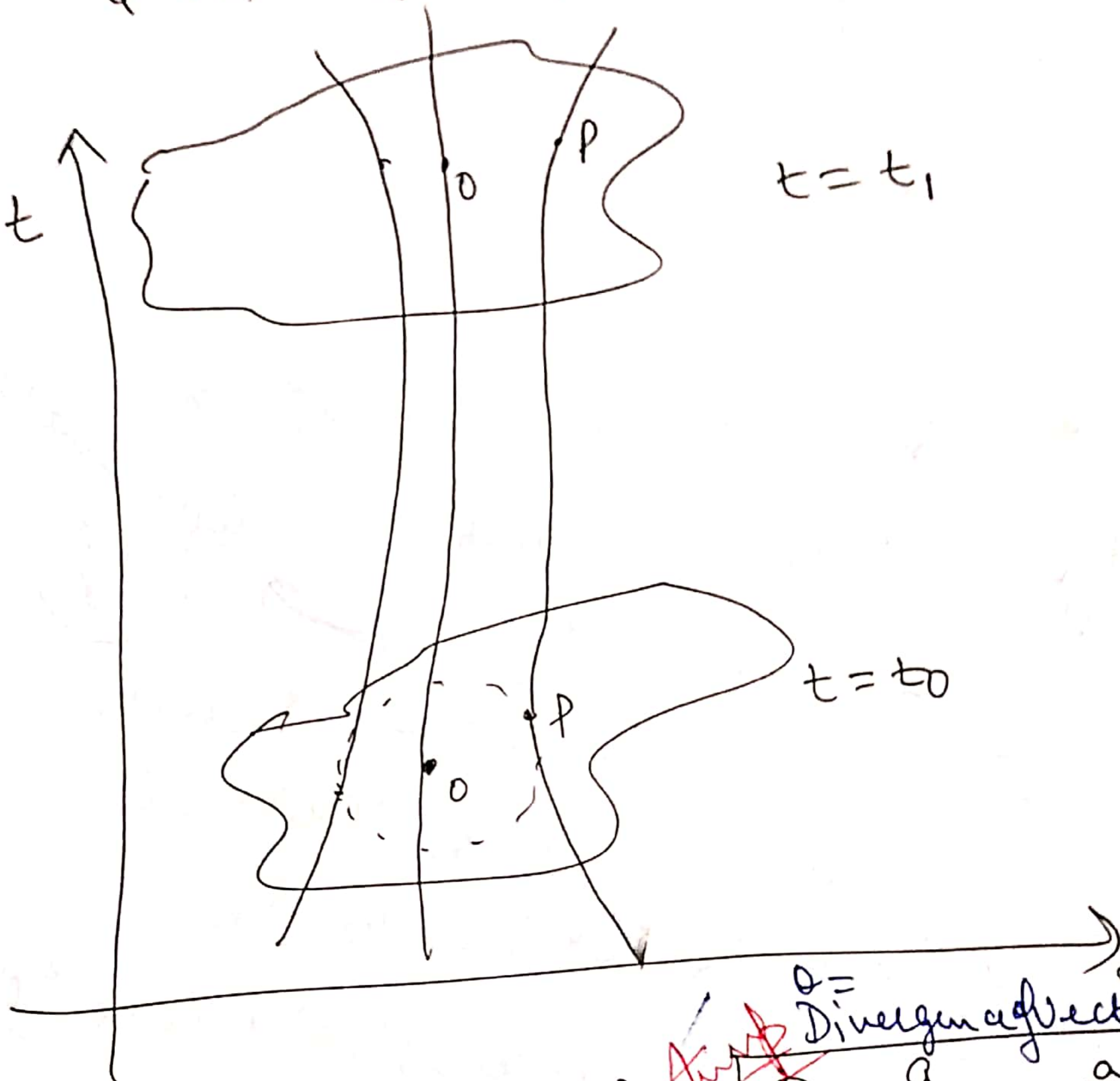
$$\xi_\varphi^\alpha(t_0+\Delta t) = \xi_\varphi^\alpha(t_0) + \left.\frac{d\xi_\varphi^\alpha}{dt}\right|_{t_0} \Delta t + O(\Delta t^2)$$
$$= \xi_\varphi^\alpha(t_0) + \xi_\varphi^k \partial_k v^\alpha(x_0)_{t_0} \Delta t$$

(29) $\therefore \xi_\varphi^a(t_1) = \xi_\varphi^a(t_0) + \Delta\xi_\varphi^a(t_0)$

where

$$\Delta\xi_\varphi^a(t_0) = B^a{}_b(t_0)\,\xi_\varphi^b(t_0)\,\Delta t$$

let $\xi_\varphi^a(t_0) = r_0(\cos\phi, \sin\phi)$



$\theta =$ Divergence of vector field

Imp

$$\boxed{\theta \equiv B^a{}_a = \partial_a v^a = \vec{\nabla}\cdot\vec{v}}$$

(30) let $B^a{}_b = \begin{pmatrix} \theta/2 & 0 \\ 0 & \theta/2 \end{pmatrix}$

$$\therefore \Delta\xi_\varphi^a = B^a{}_b(t_0)\,\xi_\varphi^b(t_0)\,\Delta t$$

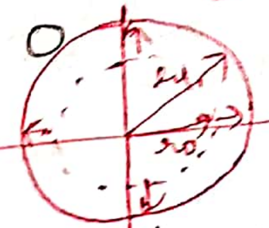
$$= \frac{\theta}{2} r_0 \Delta t\,(c\phi, s\phi)$$

$\therefore |\Delta\vec{\xi}_\varphi| = \frac{\theta}{2} r_0 \Delta t \sqrt{c^2\phi + s^2\phi} = \frac{\theta r_0 \Delta t}{2}$ = Change in Radius

$\therefore r_1 = r_0 + \frac{\theta r_0 \Delta t}{2}$ $\quad \frac{de^x_1}{dt} = \frac{\theta}{2} e^x_1 ; \frac{de^y_1}{dt} = \frac{\theta}{2} e^y_1$ 67

$$\Delta A = A_1 - A_0 = \frac{\pi \theta^2 r_0^2 \Delta t^2}{4} + \pi \theta r_0^2 \Delta t$$

as $\Delta t \to 0$



$$\Delta A = \pi \theta r_0^2 \Delta t$$

$\boxed{\theta = \frac{\Delta A}{A_0 \Delta t}}$ $\Rightarrow$ fractional change of Area per Unit time.

Expansion Parameter $\Rightarrow (\vec{\nabla} \cdot \vec{V}) = \frac{\Delta A}{A_0 \Delta t}$.

(31) $B^a_b = \partial_b v^a = \begin{pmatrix} \theta/2 & 0 \\ 0 & \theta/2 \end{pmatrix}$

Imp

It can Depend on time & choice of Reference.

$\therefore \frac{de^\alpha_t}{dt} = e^k_t \, \partial_k v^\alpha (x_0, t)$

$\Rightarrow \theta$ also depends on time & choice of Ref.

(32) In lecture $\theta = \frac{1}{V}\frac{dV}{dt}$ is derived from Mass Conservation of fluid element

(33) Shear

Shear Parameters

Let $B^a{}_b = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}$

$\xi^b_\varphi = r_0(c\phi, s\phi)$

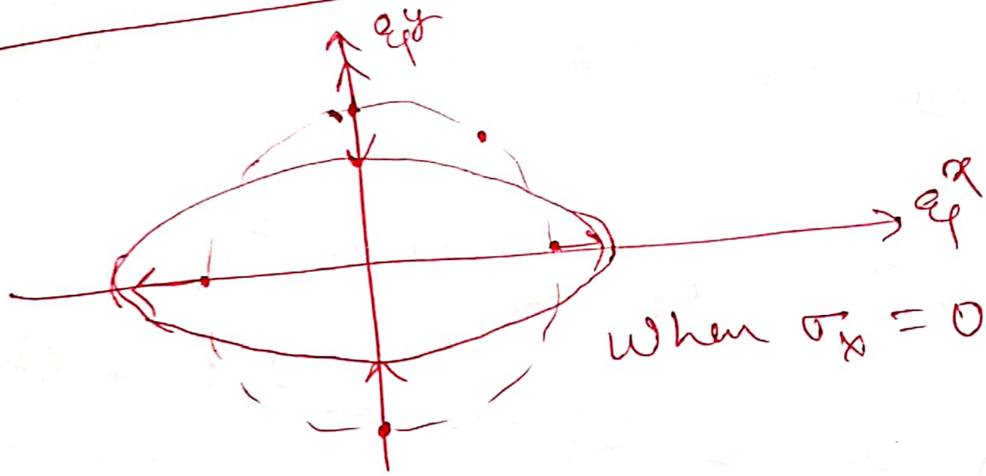
~~$\Delta\xi^a_\varphi = \dot{\xi}^k_\varphi(t_0)\partial_k \xi^a_\varphi(t_0)\Delta t$~~

$\Delta\xi^a_\varphi = B^a{}_b(t_0)\,\xi^b_\varphi(t_0)\,\Delta t$

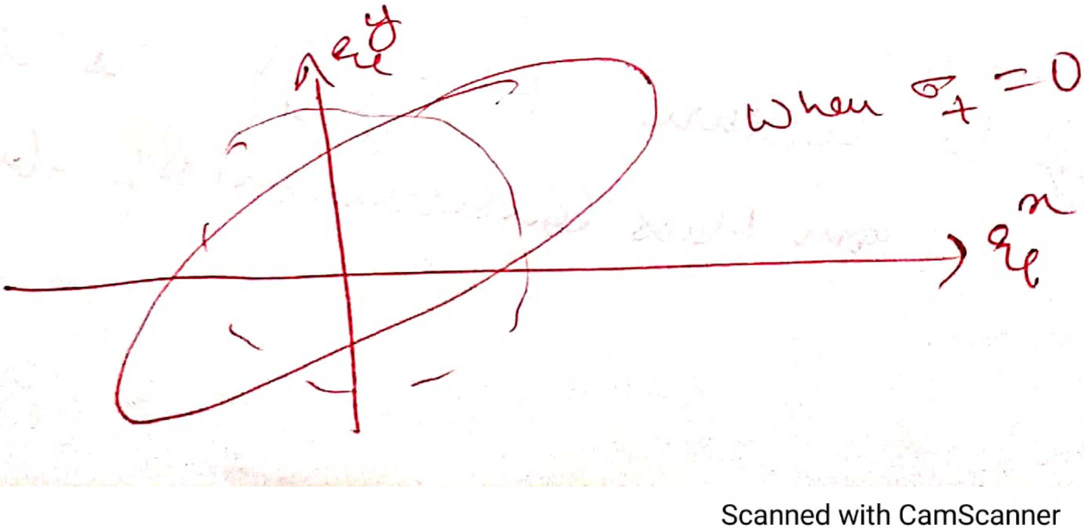
$= r_0\Delta t\left(\sigma_+\cos\phi + \sigma_\times\sin\phi,\ \sigma_\times c\phi - \sigma_+ s\phi\right)$

2Dim Shear Taken then xy Diag. element is $\dfrac{d\xi^x_\varphi}{dt} = \sigma\,\xi^x_\varphi$; $\dfrac{d\xi^y_\varphi}{dt} = -\sigma\,\xi^y_\varphi$

Rotation: $\dfrac{d\xi^x_\varphi}{dt} = \sigma\,\xi^y_\varphi$; $\dfrac{d\xi^y_\varphi}{dt} = -\sigma\,\xi^x_\varphi$



When $\sigma_\times = 0$



When $\sigma_+ = 0$

(34)

$$e_\varphi^a(t_1) = e_\varphi^a(t_0) + \Delta e_\varphi^a$$

$$e_\varphi^a(t_1) = r_0(\cos\phi, \sin\phi) + r_0\Delta t(\sigma_+ c\phi + \sigma_\times s\phi, \sigma_\times c\phi - \sigma_+ s\phi)$$

$$e_\varphi^a(t_1) = r_0(c\phi + \sigma_+\Delta t c\phi + \sigma_\times \Delta t s\phi, s\phi + \sigma_\times \Delta t c\phi - \sigma_+ \Delta t s\phi)$$

$$e_\varphi(\phi) = r_0\sqrt{1 + (\sigma_+\Delta t)^2 + (\sigma_\times \Delta t)^2 + 4\sigma_\times \Delta t s\phi c\phi + 2\sigma_+\Delta t c2\phi}$$

$$= r_0\sqrt{1 + (\sigma_+\Delta t)^2 + (\sigma_\times \Delta t)^2 + 2\sigma_\times \Delta t s2\phi + 2\sigma_+\Delta t c2\phi}$$

$$= r_0\sqrt{(1 + \sigma_+\Delta t c2\phi + \sigma_\times \Delta t s2\phi)^2}$$

Ignoring $(\Delta t)^2$ terms

$$e_\varphi(\phi) = r_0(1 + \sigma_+\Delta t c2\phi + \sigma_\times \Delta t s2\phi)$$

$$\Delta A = A_1 - A_0$$
$$= 0 \quad (\text{Ignoring } \Delta t^2 \text{ terms})$$

$\therefore$ Area Remains Same.

(35) As in (31)

β_b^a can depend on Reference pt. & time

$\Rightarrow \sigma_j^i$ can depend on time & Ref. pt.

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(26) Rotation

$\rightarrow$ Rotation Parameter

Let $B^a{}_b = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$

$$\Delta \xi^a_\varphi = \xi^k_\varphi \partial_k v^a \Delta t$$

$$= \left(r_0 (\cos\phi, \sin\phi)\right)\left(B^a_k\right)\Delta t$$

$$= r_0 \omega \Delta t \,(\sin\phi, -\cos\phi)$$

~~$r_1 = r_0 + t \Delta \xi^a_\varphi t$~~

$$\xi^a_\varphi(t_1) = \xi^a_\varphi(t_0) + \Delta \xi^a_\varphi(t_0)$$

$$= r_0(\cos\phi, \sin\phi) + r_0\omega\Delta t(\sin\phi, -\cos\phi)$$

$$\xi^a_\varphi(t_1) = r_0\left(\cos\phi + \omega\Delta t \sin\phi,\ \sin\phi - \omega\Delta t \cos\phi\right)$$

As $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\cos(\phi - \omega\Delta t) = \cos\phi \cos(\omega\Delta t) + \sin\phi \sin(\omega\Delta t)$$

$$= \cos\phi + (\sin\phi)\Delta t\,\omega$$

as Δt is small.

$$\xi^a_\varphi(t_1) = r_0(\cos\phi', \sin\phi') \qquad \phi' = \phi - \omega\Delta t$$

$\therefore$ Radius Remains Same.

$\Rightarrow$ Area Remains Same.

$\Rightarrow$ Just there is Rotation

(37) General $B^a{}_b$ can be decomposed into

Th. θ, ω, σ

Proof: see (7) L-5

(38) In 2D

$$B_{ab} = \frac{\theta \delta_{ab}}{2} + \sigma_{ab} + \omega_{ab}$$

In 3D

$$B_{ab} = \frac{\theta \delta_{ab}}{3} + \sigma_{ab} + \omega_{ab}$$

$$B_{ab} = \begin{pmatrix} \theta/3 & & 0 \\ & \theta/3 & \\ 0 & & \theta/3 \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & -(\sigma_{11}+\sigma_{22}) \end{pmatrix} + \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ & 0 & \omega_{23} \\ & & 0 \end{pmatrix}$$

(39) Expansion

As done Previously in (30)

$$\xi^a(t_0) = r_0(s\theta c\phi,\ s\theta\, s\phi,\ c\theta)$$

as $x = r\, s\theta\, c\phi$

$y = r\, s\theta\, s\phi$

$z = r\, c\theta$

$$\Delta \xi^a = B^a_b(t_0)\, \xi^b(t_0)\, \Delta t$$

$$= \frac{\theta}{3} r_0 \Delta t\, (s\theta c\phi,\ s\theta\, s\phi,\ c\theta)$$

$$|\Delta \vec{\xi}| = \frac{\theta}{3} r_0 \Delta t$$

$$\therefore r_1 = r_0 + \frac{\theta}{3} r_0 \Delta t$$

$$\frac{d\xi^a}{dt} = B^a_b\, \xi^b \Rightarrow \left\{ \begin{aligned} \frac{d\xi^x}{dt} &= \frac{\theta}{3}\xi^x \\ \frac{d\xi^y}{dt} &= \frac{\theta}{3}\xi^y \\ \frac{d\xi^z}{dt} &= \frac{\theta}{3}\xi^z \end{aligned} \right.$$

$\therefore$ Radius Increase

Sphere Remains Sphere

(40) Volume Expansion

$$\delta V = \Delta V_1 - V_0 = \pi \frac{3\theta}{3} \Delta t\, r_0^2 \left(r_0 + \frac{r_0 \Delta t\, \theta}{3} \right)$$

$$\Delta V = \tfrac{4}{3}\pi r_0^3\, \theta\, \Delta t$$

$$\Delta V = V_0\, \theta\, \Delta t$$

$$\boxed{\theta = \frac{1}{V_0}\frac{\Delta V}{\Delta t}} \Rightarrow$$ Fractional change of vol. per unit time

(41) Alternate way : Th: $\theta = \frac{1}{V_0}\frac{\Delta V}{\Delta t}$ & volume is not affected by σ & ω.

Proof: $e^a_\varphi(t_1) = (\delta^a_b + B^a_b \Delta t)\, e^b_\varphi(t_0)$

$$J = |\delta^a_b + B^a_b \Delta t|$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+3\lambda & 2\lambda \\ 0 & 1+\lambda \end{bmatrix}$$

(41) ~~Physical Meaning~~

But $\det|1+\epsilon| = 1 + Tr\,\epsilon + O(\epsilon^2)$

$(1+3\lambda)(1+\lambda) = 1+4\lambda+\cancel{3\lambda^2}$

$1 + Tr\,\epsilon = 1 + \lambda \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = 1+4\lambda$

Proof: $\ln|M| = Tr \ln M$

$$|M| = e^{Tr \ln M}$$

$$|1+\epsilon| = e^{Tr \ln(1+\epsilon)}$$

$\ln(1+\epsilon) = \epsilon$

Proof $\quad e^{\ln(1+\epsilon)} = 1 + \ln(1+\epsilon)$

$$1+\epsilon = 1 + \ln(1+\epsilon)$$

$$\ln(1+\epsilon) = \epsilon$$

$$\therefore\ |1+\epsilon| = e^{Tr\,\epsilon} = 1 + Tr\,\epsilon + O(\epsilon^2)$$

$\because \; J = |\delta^a{}_b + B^a{}_b \Delta t|$

$= 1 + Tr(B^a{}_b \Delta t)$

But $\theta = Tr\, B^a{}_b = B^a{}_a$

$\therefore \; J = 1 + \theta \Delta t$

$d^4x \longrightarrow d^4x'$

$d^4x' = J(x)\, d^4x$

$= \frac{\partial x'}{\partial x} d^4x$

This is the Coord. Transf. from $\xi^a(t_0) \to \xi^a(t_1)$

$\therefore \; V(t_1) = J\, V(t_0)$

$V_1 = (1 + \theta \Delta t) V_0$

$\theta = \frac{1}{V_0}\left(\frac{V_1 - V_0}{\Delta t}\right)$

Compare with $\theta = \frac{1}{V_0}\left(\frac{\Delta V}{\Delta t}\right)$

Only θ comes in this Transf^n which changes V

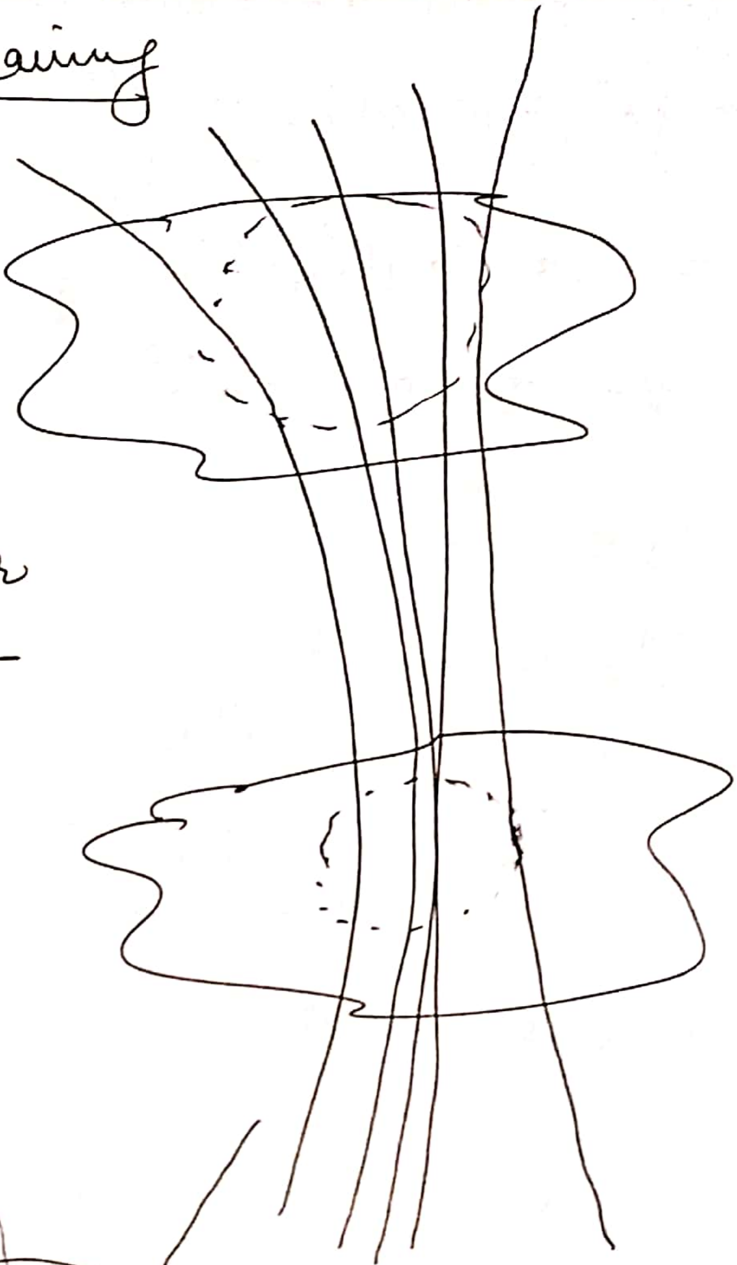
$\therefore$ σ & ω doesn't effect volume.

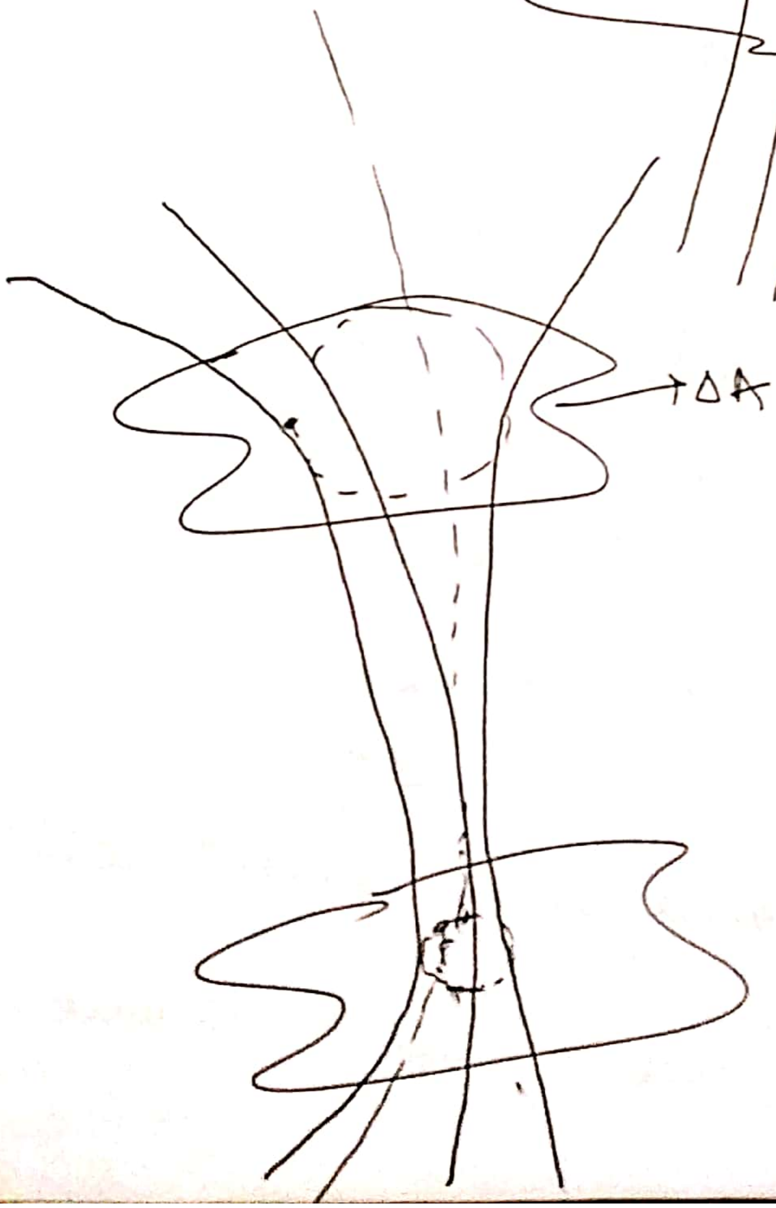
(42) Physical Meaning

θ :

$\theta = \vec{\nabla} . \vec{v}$

= Divergence of vector field





σ:

ω:

(43) Previously all we have done in Euclidean 3D flat space.

(44) Why not treat ξ_ℓ as ρ?

$$\frac{d\xi_\ell^\alpha}{dt}(t) = v^\alpha(x_0 + \xi_\ell, t) - v^\alpha(x_0, t)$$
$$= \xi_\ell^k \partial_k v^\alpha(x_0, t) + O(\xi_\ell^2)$$

$$\xi_\ell(t_0 + \Delta t) = \xi_\ell(t_0) + \left.\frac{d\xi_\ell}{dt}\right|_{t_0} \Delta t + O(\Delta t^2)$$

$$\xi_\ell(t_0 + \Delta t) = \xi_\ell(t_0) + \xi_\ell^k \partial_k v(x_0, t_0)\,\Delta t.$$

$$d\rho = \rho(t + dt, x + dx) - \rho(t, x)$$
$$d\xi_\ell = \xi_\ell(t_0 + dt, x_0 + dx) - \xi_\ell(t_0, x_0)$$

$$\frac{d\xi_\ell}{dt} = \left.\frac{\partial \xi_\ell}{\partial t}\right|_{t_0 x_0} + \left.(\vec{\nabla}\vec{\xi}_\ell)\cdot \vec{dx}\right|_{t_0 x_0}$$

---

(46) Longruences of Timelike ~~long~~ geodesics is concerned.

$u^a u_a = 1$; $u^\beta \nabla_\beta u^\alpha = 0$; $\alpha_u \xi^\alpha = \alpha_u u^\alpha = 0$;

$\xi^\beta \nabla_\beta u^\alpha = u^\beta \nabla_\beta \xi^\alpha$; $u^\alpha \xi_\alpha = 0$

(47) $g^i_j = g_{j\alpha} g^{\alpha i}$

$g_{j\alpha} \overset{*}{=} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \\ & & & -1 \end{pmatrix}$ $g^{\alpha i} \overset{*}{=} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \\ & & & -1 \end{pmatrix}$

as $A^{-1} = \dfrac{\text{adj} A}{|A|}$

$\text{adj} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \\ & & & -1 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \\ & & & 1 \end{pmatrix}$

$|A| = -1$

$\therefore A^{-1} = A \Rightarrow g_{j\alpha} \overset{*}{=} g^{\alpha i}$

$\therefore$ ~~$g^i_j \overset{*}{=}$~~ $\delta^i_j \overset{*}{=} g_{j\alpha} g^{\alpha i}$

But this is Tensorial

$\therefore \delta^i_j = g_{j\alpha} g^{\alpha i}$

But By Def. $g^i_j = g_{ja} g^{ai}$

$\therefore \underline{g^i_j = \delta^i_j}$

(48) $g_{\alpha\beta}$ can be decomposed into Longitudinal part $u_\alpha u_\beta$ & Transverse part $h_{\alpha\beta}$

$$g_{\alpha\beta} = h_{\alpha\beta} + u_\alpha u_\beta$$

see L-5 (37)

(49) Properties $h_{\alpha\beta}$

$$h^a_{\ i} h^i_{\ b} = h^a_{\ b}$$

$$u^a h_{ab} = h_{ab} u^b = 0$$

see L-5 (33)

$$h^i_{\ i} = +3$$

$$h^a_{\ b} = g^a_{\ b} - u^a u_b$$

$$h^a_{\ b} \overset{*}{=} g^a_{\ b} - u^a u_b$$

$$Tr(h^a_{\ b}) \overset{*}{=} Tr(g^a_{\ b}) - Tr(u^a u_b)$$

$$\overset{*}{=} Tr(\delta^a_b) - Tr(u^a u_b)$$

$$\overset{*}{=} 4 - 1 \overset{*}{=} 3$$

$$\boxed{h^i_{\ i} \overset{*}{=} 3}$$

$$h^i_{\ i} \overset{*}{=} g^i_{\ i} - u^i u_i \overset{*}{=} \delta^i_i - u^i u_i$$

$$\eta^{ij} h_{ij} \overset{*}{=} 3$$

$$0 - h_{11} - h_{22} - h_{33} \overset{*}{=} 3$$

$$\boxed{h_{ii} \overset{*}{=} -3}$$

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$h_{i}{}^{i} = -3$

$h^{\alpha\beta}\sigma_{\alpha\beta} = 0$

Proof: $h^{\alpha\beta}\left(B_{(\alpha\beta)} - \frac{h_{\alpha\beta}\theta}{3}\right)$

$h^{\alpha\beta} B_{(\alpha\beta)} = h^{\alpha\beta}\left(B_{\alpha\beta} - B_{[\alpha\beta]}\right) = \theta - \cancel{h^{\alpha\beta}B_{[\alpha\beta]}} \to 0$

$h^{\alpha\beta}$ is sym.

$h^{\alpha\beta} = g^{\alpha\beta} - u^{\alpha}u^{\beta}$

$\therefore h^{\alpha\beta} B_{(\alpha\beta)} = \theta \Rightarrow \theta - \theta = h^{\alpha\beta}\sigma_{\alpha\beta} = 0$

(2) $h^{\alpha\beta}\omega_{\alpha\beta} = 0$

Proof ?
0.

or Alternatively

$h^{\alpha\beta}\sigma_{\alpha\beta} = 0$ go to 2.I.J.

Think $h^{\alpha\beta} = \delta^{\alpha\beta} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$\therefore h^{\alpha\beta}\sigma_{\alpha\beta} = Tr(\sigma) = 0$

$\therefore \; h_{ii} = -3$

$\boxed{\text{Is Trace a tensorial eqn ?}}$

(50) Evolution of Deviation vector

$$\frac{D\xi^{\alpha}}{dt} = u^{\beta}\nabla_{\beta}\xi^{\alpha} = \xi^{\beta}\nabla_{\beta}u^{\alpha} = \xi^{\beta}B^{\alpha}_{\beta}$$

B^{α}_{β} measures the failure of ξ^{α} to be $\parallel$ transported

$B_{\alpha\beta} = \nabla_{\beta}u_{\alpha}$ is purely spatial see L-5 (41)

(51) $\therefore \; B_{\alpha\beta} = \theta \dfrac{h_{\alpha\beta}}{3} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$

$\theta = h^{\alpha\beta}B_{\alpha\beta}$ as $h^{\alpha\beta}\sigma_{\alpha\beta} = h^{\alpha\beta}\omega_{\alpha\beta} = 0$

see (45) L-5

(52) Comparing

$$\frac{d\xi^{\alpha}}{dt} \;\&\; \frac{D\xi^{\alpha}}{dt}$$

54

Assuming ~~congruences~~ Timelike Geod

$\therefore u^\beta \nabla_\beta u^\alpha = 0$

$u^\alpha \xi_{\varphi\alpha} = 0$

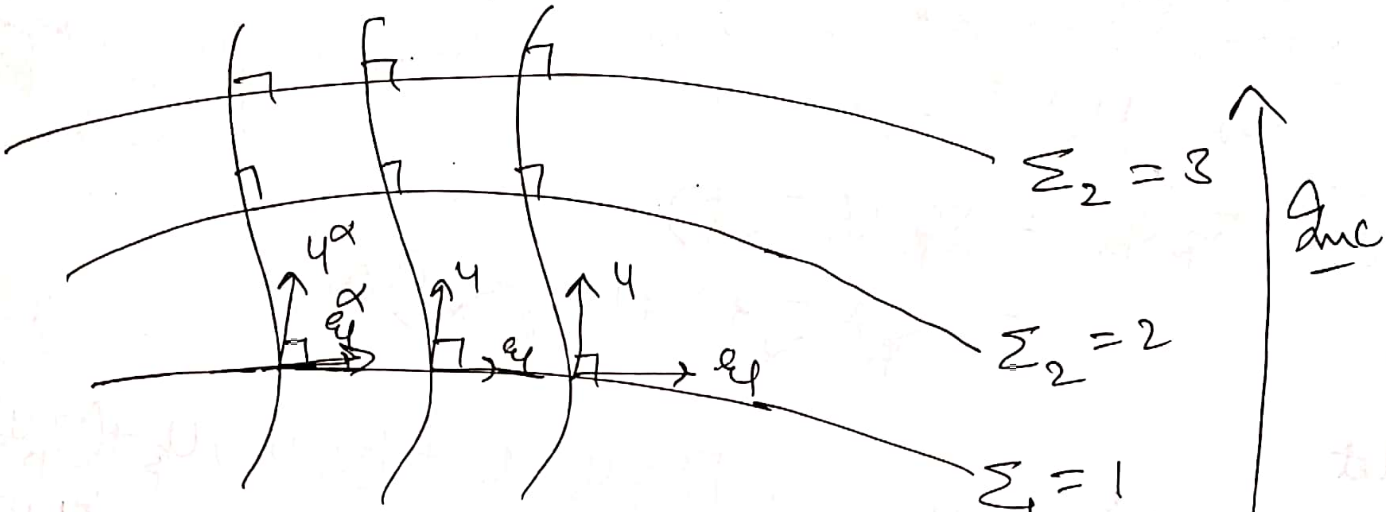
$u^\alpha u_\alpha = 1$

→ Time like Geod. is not Needed

$\mathcal{L}_u \xi_\varphi^\alpha = \mathcal{L}_{\xi} u^\alpha = 0$

$u^\beta \nabla_\beta \xi_\varphi^\alpha = \xi_\varphi^\beta \nabla_\beta u^\alpha$

Hyper surf ⇒ $\phi(x^i) = c$



$$\frac{D\xi_\varphi^\alpha}{dt} = u^\beta \nabla_\beta \xi_\varphi^\alpha = \xi_\varphi^\beta \nabla_\beta u^\alpha = \xi_\varphi^\beta B_\beta^\alpha$$

As ξ_φ lies on the Hyper Surface

$B_{\alpha\beta}$ tells how ξ_φ changes on the Hyper Surf.

$B_{\alpha\beta} = \nabla_\beta u_\alpha$ But $u_\alpha = \vec{C} n_\alpha = C(B\partial_\alpha \phi)$ (const & const)

$= A\partial_\alpha \phi$

See L-6 9

As $\omega_{\alpha\beta}$ is Spatial

$\therefore \partial_\beta A = (u^\alpha \partial_\alpha A) u_\beta$ ⇒ A must be const on each H.S.

It varies only in the direction Normal to H.S

$\Rightarrow \omega_{\alpha\beta} = 0$

for Time like Geod. Only

(55) The congruence will be Hyp. Orth. if $u^\alpha \,\&\, n^\alpha$

$\therefore\; u_\alpha = A\,\partial_\alpha \phi \Rightarrow \phi$ inc. towards future

$\phi(x^i) = c$

A +ve

A can be found by Normalization Condn

$$U_\alpha U^\alpha = 1$$

(56) $$\nabla_\beta U_\alpha = \nabla_\beta (A \nabla_\alpha \phi)$$

$$= \nabla_\beta A \nabla_\alpha \phi + (\nabla_\beta \nabla_\alpha \phi) A$$

Let

$$\nabla_{[\beta} u_\alpha u_{\gamma]} = \frac{1}{3!}\left[\begin{array}{l}(\nabla_\beta U_\alpha) U_\gamma + (\nabla_\alpha U_\gamma) U_\beta + (\nabla_\gamma U_\beta) U_\alpha \\ -(\nabla_\alpha U_\beta) U_\gamma - (\nabla_\beta U_\gamma) U_\alpha - (\nabla_\gamma U_\alpha) U_\beta\end{array}\right]$$

$$\nabla_\beta u_\alpha = \partial_\beta A\, \partial_\alpha \phi + (\nabla_\beta (\partial_\alpha \phi))\, A$$

~~Imp~~ As $\phi'(x') = \phi(x)$ (Scalar fn)

$d\phi' = d\phi$ $\therefore\; d\phi$ is also scalar fn

$d\phi$ is scalar not $\partial_\alpha \phi$

But $\nabla_\beta \phi = \partial_\beta \phi$ ✗

$$\therefore\; \nabla_\beta (\partial_\alpha \phi) = \partial_\beta (\partial_\alpha \phi)$$

$$\therefore\; \nabla_\beta U_\alpha = \partial_\beta A\, \partial_\alpha \phi + (\partial_\beta \partial_\alpha \phi) A$$

By $\partial_\alpha A\, \partial_\beta \phi = \partial_\beta A\, \partial_\alpha \phi \Rightarrow \nabla_{[\beta} u_\alpha u_{\gamma]} = 0$

(57) Cong. (T/N/S & not Nec. geodesic)

Th: if the Hyper Surf. Orth. then $\nabla_{[\beta} u_\alpha u_{\gamma]} = 0$

Converse: if $\nabla_{[\beta} u_\alpha u_{\gamma]} = 0$ then Cong. is Hs. orth.

i.e. $u_\alpha \propto \partial_\alpha \phi$.

(58) In the derivation we never used the fact that congruence is Timelike nor it is Geodesic. Infact Congruence can be Time/Null/Spacelike.

(59) Nor Did we use Normalization Condition

$$u^\alpha u_\alpha = -1$$

$$n^\alpha n_\alpha = 1$$

(60) Frobenius Theorem

Congruence of Curves (T/N/S) & not Necessarily Geod. is Hyp. S. Orth. iff $\omega_{\alpha\beta} = 0$

$\nabla_{[\beta} u_\alpha u_{\gamma]} = 0$

(61) $$B_{\alpha\beta} = \frac{B_{\alpha\beta} - B_{\beta\alpha}}{2} + \frac{B_{\alpha\beta} + B_{\beta\alpha}}{2}$$

$$= \frac{B_{[\alpha\beta]}}{2} + \frac{B_{(\alpha\beta)}}{2}$$

from (57) $3 \nabla_{[\beta} u_\alpha u_{\gamma]} = u_\gamma B_{[\alpha\beta]} + u_\beta B_{[\gamma\alpha]} + B_{[\beta\gamma]} u_\alpha$

$$0 = \omega_{\alpha\beta} u_\gamma + \omega_{\gamma\alpha} u_\beta + \omega_{\beta\gamma} u_\alpha$$

$B_{[\alpha\beta]} = \omega_{\alpha\beta}$ (using Timelike Geod.)

Multiply u^γ $\quad 0 = \omega_{\alpha\beta} + \omega_{\gamma\alpha} u_\beta u^\gamma + \omega_{\beta\gamma} u_\alpha u^\gamma$

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But $u_\alpha u^\gamma = u^\gamma u_\alpha$ — only for timelike Geod

$\therefore\ \omega_{\alpha\beta} = 0$ as $\omega_{\alpha\beta} u^\beta = 0$

$\therefore$ for Timelike Geod. H.S. Orth $\Rightarrow \omega_{\alpha\beta} = 0$

(62) for Direct Proof from Time like Geod see (54)

(63) Th: $u_\alpha = A\, \partial_\alpha \phi \Rightarrow$ For H.S. Orth. curves

Doubt — if $\exists\ \varphi$ s.t. $\varphi = \int A(\phi)\, d\phi$ then the curve satisfies Geod. Eqn.

Proof: Let $\exists\ \varphi$ s.t. $\varphi = \int A(\phi)\, d\phi$

$$\partial_\phi \varphi = A$$

$$\therefore\ u_\alpha = (\partial_\alpha \phi)\, \partial_\phi \varphi$$

$$u_\alpha = \partial_\alpha \varphi$$

$$u^\beta \nabla_\beta u^\alpha = \partial^\beta \varphi\, (\nabla_\beta \partial_\alpha \varphi)$$

$$= \partial^\beta \varphi\, \partial_\alpha \partial_\beta \varphi = \frac{(\partial_\alpha \partial^\beta \varphi)\partial_\beta \varphi + \partial^\beta \varphi\, \partial_\alpha \partial_\beta \varphi}{2}$$

$$= \frac{\partial_\alpha (\partial^\beta \varphi\, \partial_\beta \varphi)}{2}$$

$$= \frac{\partial_\alpha (u^\beta u_\beta)}{2}$$

for ~~any~~ Null Curve

$$u^\beta u_\beta = \frac{dx^\beta\, dx_\beta}{(d\lambda)^2} = \frac{0}{(d\lambda)^2} = 0$$

for time like Curve

$$u^\beta u_\beta = \frac{dx^\beta\, dx_\beta}{(d\tau)^2} = -1$$

for space like Curve $u^\beta u_\beta = dx^\beta dx_\beta = +$

~~By Continuity for Null Curve~~

$\therefore$ for Any Curve $\partial_\alpha (u^\beta u_\beta) = 0$

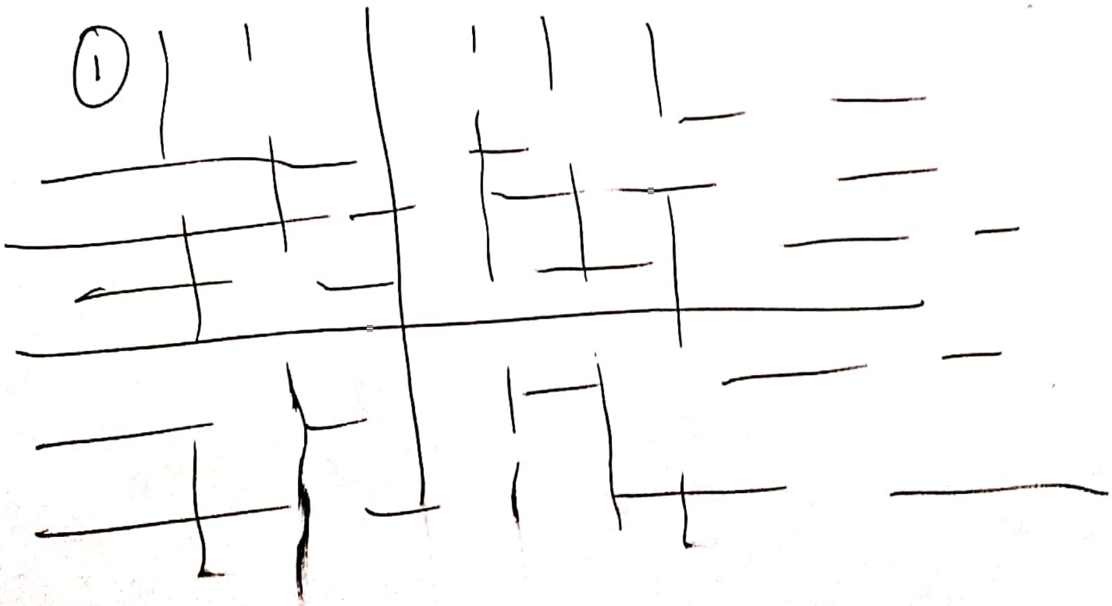
$\therefore$ $u^\beta \nabla_\beta u^\alpha = 0 \Rightarrow$ Geod. eqn

(64) Inverse of (63) ?

(66) Inverse of Frobenius Th.

(67) Physical Meaning when Taken Example in SR

(1)



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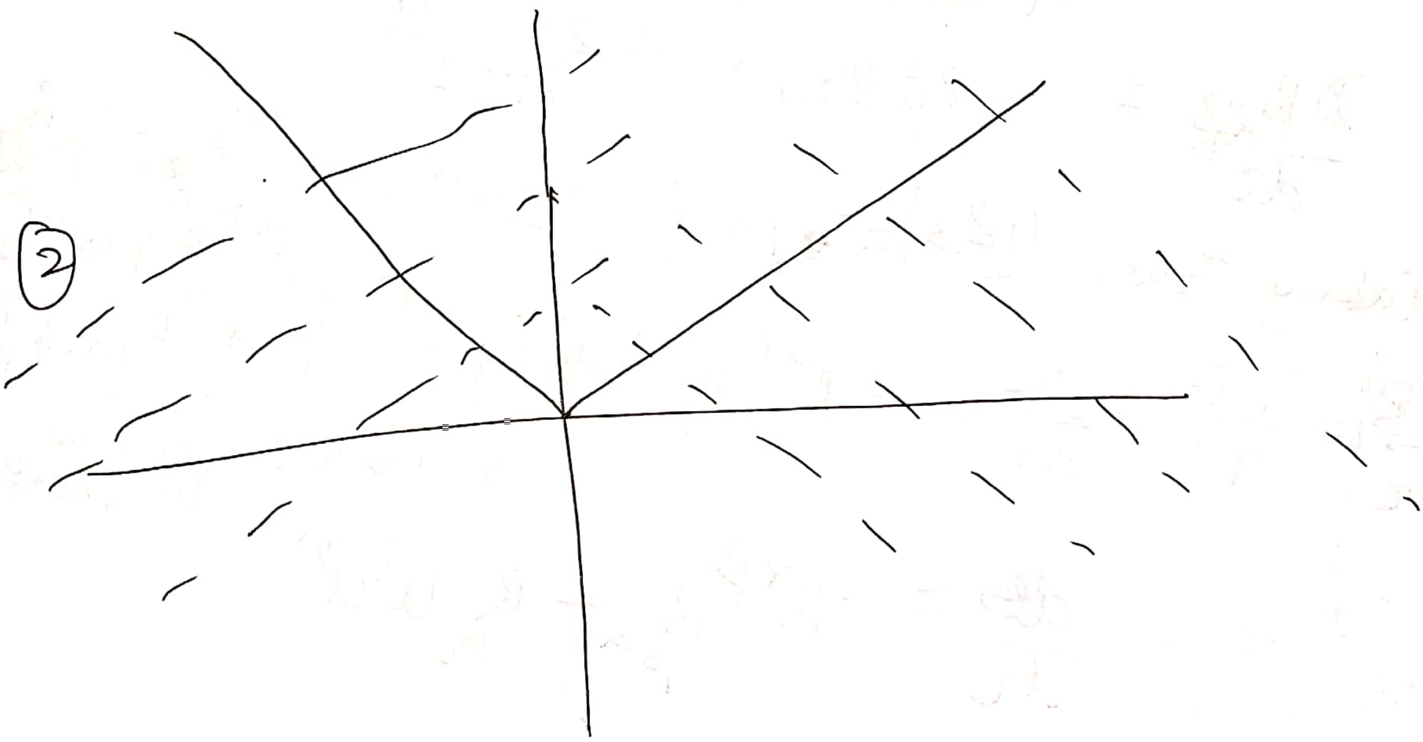


(68) Raychanduri eqn for timelike geod

$$\cancel{\frac{DB_{\alpha\beta}}{d\tau} = -B_{\alpha i}B^i_\beta - R_{\alpha i \beta j}u}$$

$$\frac{DB_{\alpha\beta}}{d\tau} = u^i \nabla_i (\nabla_\beta u_\alpha)$$
$$= u^i(\nabla_\beta \nabla_i u_\alpha + R_{\alpha r i \beta} u^r)$$

$$\cancel{= \nabla_i \nabla_\beta - \nabla_i}$$

$$\frac{DB_{\alpha\beta}}{d\tau} = -B^i_\beta B_{i\alpha} + u^i R_{\alpha r i \beta} u^r$$

Taking Trace $(1\&3) = -(1\&4) = -(2\&3) \Rightarrow$ $\cancel{g^{\alpha r} R_{\beta\alpha r\delta} = R_{\beta\delta}}$

$$\boxed{\begin{array}{l} g^{\alpha r} R_{\alpha\beta r\delta} = R_{\beta\delta} \\ g^{\alpha r} R_{\beta\alpha r\delta} = -R_{\beta\delta} \end{array}}$$

$$\frac{DB^i_i}{d\tau} = \frac{D\theta}{d\tau} = \frac{d\theta}{d\tau} = -B^{\alpha\beta}B_{\beta\alpha} - \underbrace{u^i R_{ri} u^r}$$

Scalar $\therefore$ can change position

$$\frac{d\theta}{d\tau} = -B^{\alpha\beta}B_{\beta\alpha} - R_{ri}u^i u^r$$

(69) $B^{\alpha\beta}B_{\beta\alpha} = +\frac{\theta^2}{3} + \sigma^{\alpha\beta}\sigma_{\beta\alpha} - \omega^{\alpha\beta}\omega_{\beta\alpha}$

L-6 (19)

(70) $$\boxed{\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} + \sigma^{\alpha\beta}\sigma_{\beta\alpha} + \omega^{\alpha\beta}\omega_{\beta\alpha} - R_{ri}u^i u^r}$$

As $\sigma^{\alpha\beta}$ is spatial

Focusing theorem

(72) Let the cong. of timelike geod be H.S.Orth. $\therefore \omega_{\alpha\beta} = 0$ & let strong Energy condition hold

$$R_{\alpha\beta}u^{\alpha}u^{\beta} \geq 0 \Rightarrow \frac{d\theta}{d\tau} \leq 0$$

(73) **Physical Interpretation**

Gravitation is an attractive force when strong Energy condition holds.

(74)

Caustic: A point at which some of the geodesics come together

A Caustic is singularity of Congruence & earlier we assumed that definition of Congruence is unique curves passing through each event.

$\therefore$ Raychaudhuri Eqn. is not valid at such points.

Only valid before Caustics happen.

(75) Under Condn of focusing theorem $\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{3}$

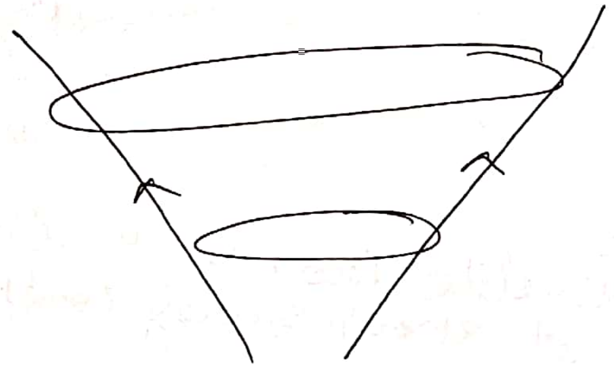
$$\frac{d\theta}{\theta^2} \leq -\frac{d\tau}{3} \Rightarrow \theta^{-1}(\tau) - \theta^{-1}(0) \geq \frac{\tau}{3}$$

Initially θ

Let Initially converging $\therefore \theta(0) < 0$

$$\frac{1}{\theta(\tau)} \geq \frac{\tau}{3} + \frac{1}{\theta(0)}$$

76

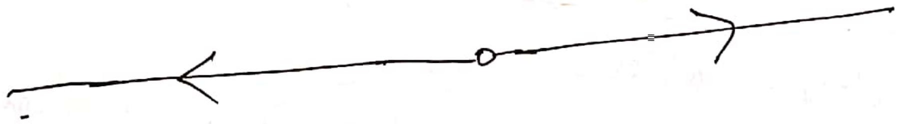


$\theta = \frac{1}{A_0}\frac{dA}{dt}$ +ve

$\therefore$ Div. +ve

$\frac{dA}{dt} \to \infty$

$\theta \to \infty$



Similarly

$\frac{dA}{dt} \to -\infty$

$\frac{dA}{dt}$ -ve.

$\therefore \theta \to -\infty$ at Caustic

77 Proper Time to get to Caustic ?

$$\frac{1}{\theta(\tau)} \geq \frac{\tau}{3} + \frac{1}{\theta_0}.$$

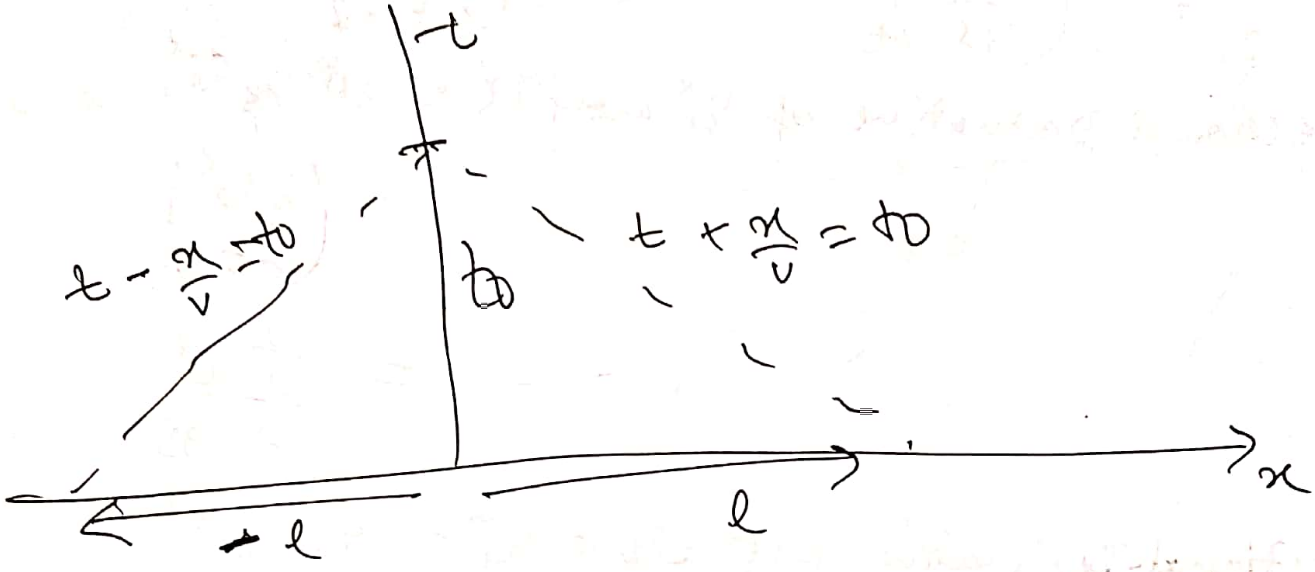
$\theta(\tau) \to -\infty$

$$0 \geq \frac{\tau}{3} + \frac{1}{\theta_0}$$

$$\boxed{\tau \leq \frac{3}{|\theta_0|}}$$

Congruence will Become Caustic with propertime $\leq \frac{3}{|\theta_0|}$

(78) Use $\tau \leq \frac{3}{|\theta_0|}$ in SR



(79) Why our formalism breaks down at Caustic? $\frac{7}{10}$

Ans $\quad \xi^\alpha = \left(\frac{\partial x^\alpha}{\partial s}\right)_t \qquad u^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_s$

Directional Derivative of ξ^α along $u = u^\beta \partial_\beta \xi^\alpha$

$$= \left(\frac{\partial \xi^\alpha}{\partial t}\right)_s$$

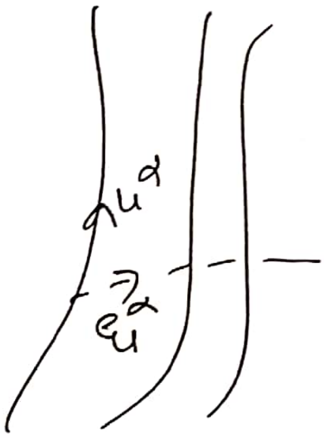
$$= \frac{\partial^2 x^\alpha}{\partial t \partial s}$$

Directional Derivative of u^α along $\xi^\alpha = \xi^\beta \partial_\beta u^\alpha$

$$= \left(\frac{\partial u^\alpha}{\partial s}\right)_t$$

$$= \frac{\partial^2 x^\alpha}{\partial s \partial t}$$

$$\mathcal{L}_{\xi} u^\alpha = \xi^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta \xi^\alpha = 0 = -\mathcal{L}_u \xi^\alpha$$



But at Caustic this formalism won't work as at a pt. Both Directional deriv. doesn't $\exists$.

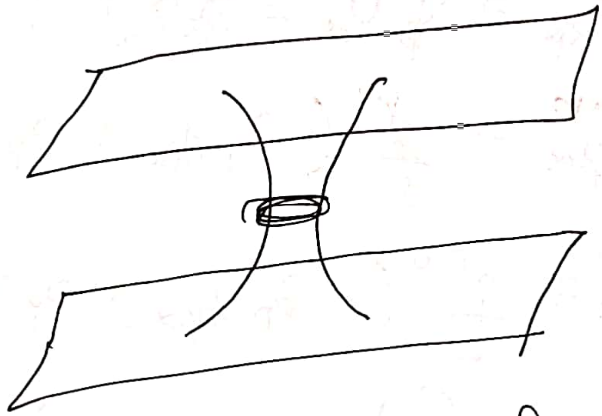
$$\frac{D\xi^\alpha}{d\lambda} = u^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta u^\alpha = \xi^\beta B^\alpha{}_\beta$$

Not Valid

91

(80) From seeing the spacetime Diagram we can say if the specific Energy condition is being followed or not.

Example wormhole.



Example : Our Universe Being Accelerated

(81) Relation of Cosmological constant with violation with strong Energy Cond^n.

(82) let the metric be

$$ds^2 = dt^2 - a^2(t)\,(dx^2 + dy^2 + dz^2)$$

$$B_{\alpha\beta} = \nabla_\beta U_\alpha = \partial_\beta \partial_\alpha t + \Gamma^{r}_{\beta\alpha} U_r$$

$$\Gamma^{r}_{\beta\alpha} = \frac{g^{ri}}{2}\left(-\partial_i g_{\beta\alpha} + \partial_\beta g_{\alpha i} + \partial_\alpha g_{i\beta}\right)$$

$$\Gamma^{0}_{\beta\alpha} = \frac{g^{00}}{2}\left(-\partial_0 g_{\beta\alpha} + \partial_\beta g_{\alpha 0} + \partial_\alpha g_{\beta 0}\right)$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$$

$$g^{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & \frac{1}{-a^2} & & \\ & & \frac{1}{-a^2} & \\ & & & \frac{1}{-a^2} \end{pmatrix}$$

$$\Gamma^{0}_{\beta\alpha} = \frac{1}{2}\left(-\partial_t g_{\beta\alpha}\right)$$

$$\Gamma^{j}_{\beta\alpha} = \frac{g^{ji}}{2}\left(-\partial_i g_{\beta\alpha} + \partial_\beta g_{\alpha i} + \partial_\alpha g_{i\beta}\right)$$

$$\Gamma^{j}_{\beta\alpha} = \frac{+g^{ji}}{2\cancel{a}}\left(\partial_\beta g_{\alpha i} + \partial_\alpha g_{i\beta}\right) = \frac{-1}{2a^2}\left(\partial_\beta g_{\alpha i} + \partial_\alpha g_{i\beta}\right)$$

$$B_{\alpha\beta} = \cancel{\partial_\beta \partial} \; \Gamma^{0}_{\beta\alpha} U_0 + \cancel{\Gamma^{i}_{\beta\alpha} U_i}$$

$$= -\frac{\partial_t g_{\beta\alpha}}{2}$$

$$h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$$

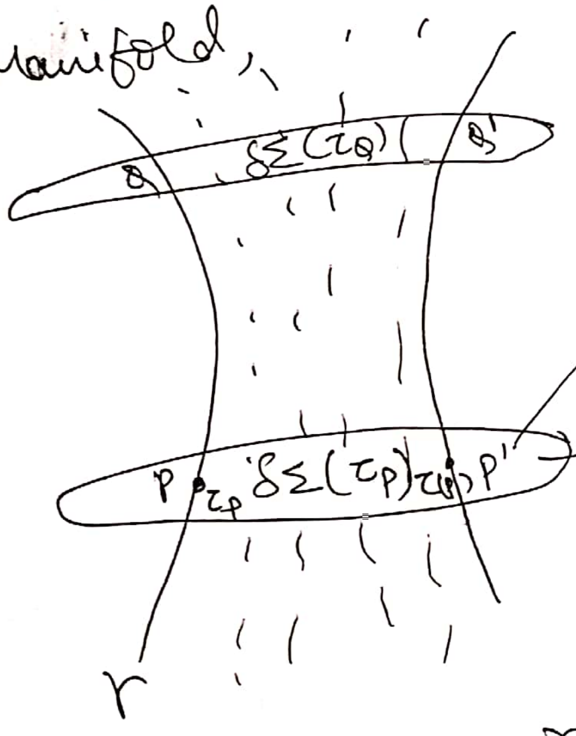
(84) Till Now, in flat spacetime

① By fluid Mech
② By Jacobian
③ Individual

All Euclidean

$$\theta = \frac{1}{\delta V}\frac{d(\delta V)}{d\tau}$$

(85) In Manifold,



$y^a \to$ Intrinsic coordinates

$\to$ set of all $\tau = \tau_P$ (Hypersurface of $\tau = \tau_P$)

⇒ Assuming curve γ is $\perp$ to H.S.

⇒ Now on this H.S let the coord. be y^a $(a = 1, 2, 3)$

⇒ Assuming Each geodesic has same y^a along it

∴ On H.S $\delta\Sigma(\tau_Q)$ y^a we can know by y^a on $\delta\Sigma(\tau_P)$

$\therefore$ we have a new coordinate system ...

But old coord. system is x^α $(a = 0, 1, 2, 3)$

$\therefore \exists$ Transformation

$y^a \to$ Intrinsic coordinates

$$x^\alpha = f^\alpha(\tau, y^a)$$

$x^\alpha \to$ Global coordinates

$$x^\alpha = f^\alpha(y^a)$$

Parametric eqn

Eg. 2D sphere in 3D

$f^1(\theta, \phi) = x$

$f^2(\theta, \phi) = y$

$f^3(\theta, \phi) = z$

$$u^\alpha = \left(\frac{dx^\alpha}{d\tau}\right)$$

$$dx^\alpha = \left(\frac{\partial f^\alpha}{\partial \tau}\right)_{y^a} d\tau + \left(\frac{\partial f^\alpha}{\partial y^a}\right)_\tau dy^a$$

Along geod. y^a is const.

$$\therefore \frac{dx^\alpha}{d\tau} = u^\alpha = \left(\frac{\partial f}{\partial \tau}\right)_{y^a}.$$

(86) $e^\alpha_a = \frac{dx^\alpha}{dy^a}$ on H.S. where τ is const.

Compare $\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial s}\right)_t$

$$dx^\alpha = \left(\frac{\partial f^\alpha}{\partial y^a}\right)_\tau dy^a$$

$$e^\alpha_a = \left(\frac{\partial f^\alpha}{\partial y^a}\right)_\tau \quad \text{Tangents on the H.S.}$$

(87) Only on γ by assumption we have

$$e^\alpha_a u_\alpha = 0$$

Compare $\mathcal{L}_\xi u^\alpha = \mathcal{L}_u \xi^\alpha = 0.$

(88) $$\mathcal{L}_u e^\alpha_a = u^\beta \nabla_\beta e^\alpha_a - e^\beta_a \nabla_\beta u^\alpha$$

$$= \frac{\partial x^\beta}{\partial \tau} \frac{\partial}{\partial x^\beta} e^\alpha_a - \frac{\partial x^\beta}{\partial y^a} \frac{\partial u^\alpha}{\partial x^\beta}$$

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$$= \frac{\partial^2 x^\alpha}{\partial \tau \partial y^a} - \frac{\partial^2 x^\alpha}{\partial y^a \partial \tau} = 0$$

(89) Definition: 3-Tensor

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$$

(90) Th: h_{ab} is Tensor w.r.t. transfn $y^a \longrightarrow y^{a'}$
& scalar w.r.t. Transfn $x^\alpha \longrightarrow x^{\alpha'}$

Proof: (1) $h_{a'b'} \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^{b'}}{\partial y^b}$

$$= g_{\alpha\beta} e^\alpha_{a'} e^\beta_{b'} \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^{b'}}{\partial y^b}$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\cancel{\partial y^{a'}}} \frac{\partial x^\beta}{\cancel{\partial y^{b'}}} \frac{\cancel{\partial y^{a'}}}{\partial y^a} \frac{\cancel{\partial y^{b'}}}{\partial y^b}$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} = g_{\alpha\beta} e^\alpha_a e^\beta_b$$

$$h_{a'b'} \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^{b'}}{\partial y^b} = h_{ab} \qquad \therefore h_{ab} \text{ is Tensor}$$

(2). $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b = g_{\alpha'\beta'} \partial_\alpha x^{\alpha'} \partial_\beta x^{\beta'} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b}$

$$h_{ab} = g_{\alpha'\beta'} \frac{\partial x^{\alpha'}}{\partial y^a} \frac{\partial x^{\beta'}}{\partial y^b}$$

$\therefore h_{ab}$ is scalar w.r.t. $x^\alpha \longrightarrow x^{\alpha'}$

(91) **Theorem**

h_{ab} is metric Tensor on H.S.

Proof: $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial \tau}\right)_{y^a} d\tau + \left(\frac{\partial x^\alpha}{\partial y^a}\right)_\tau dy^a$$

$$ds^2\Big|_{H.S.} = g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a}\right)_\tau \left(\frac{\partial x^\beta}{\partial y^b}\right)_\tau dy^a dy^b$$

$$= g_{\alpha\beta}\, e^\alpha_a e^\beta_b\, dy^a dy^b$$

$$ds^2 = h_{ab}\, dy^a dy^b$$

$\therefore$ h_{ab} is 3 metric Tensor.

$g_{\alpha\beta} = \begin{pmatrix} & \\ & h_{ab} \end{pmatrix}$

(92) $h_{ab} = g_{\alpha\beta}\, e^\alpha_a e^\beta_b$

$= (h_{\alpha\beta} + U_\alpha U_\beta)\, e^\alpha_a e^\beta_b$

$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & h_{ab} \end{pmatrix}$

But only on Γ $\quad U_\alpha e^\alpha_a = 0$

$\therefore$ $\boxed{h_{ab} = h_{\alpha\beta}\, e^\alpha_a e^\beta_b \text{ on } \Gamma.}$

$g_{\alpha\beta} = h_{\alpha\beta} + U_\alpha U_\beta$

(93) **Inverse**: $h^{ab} h_{bc} = \delta^a_c$

$h^{ca} h_{ab} = h^{ca} h_{\alpha\beta}\, e^\alpha_a e^\beta_b \rightarrow$ on Γ

$\delta^c_b\, h^{\gamma\alpha} = h^{ca}\, \delta^\gamma_\beta\, e^\alpha_a e^\beta_b$

$\boxed{h^{\beta\alpha} = h^{ba}\, e^\alpha_a e^\beta_b \quad \text{on } \Gamma \text{ only}}$

$\hookrightarrow$ $h^{\beta\alpha}$ decomposed into $e^\alpha_a e^\beta_b$.

(94) 98

$$h_{ab} = \partial_a y^{a'} \partial_b y^{b'} h_{a'b'}$$

$$y^{a'} \longrightarrow y^a$$

$$d^3 y^{a'} = J(y)\, d^3 y$$

$$h = J^2(y)\, h'$$

Only for timelike or spacelike. Depending on if H.S is S or T.

for S|T $\quad d^3 y' = \sqrt{\frac{|h|}{|h'|}}\, d^3 y \Rightarrow \underline{\sqrt{|h'|}\, d^3 y' = \sqrt{|h|}\, d^3 y} = d\Sigma$

← Directed surface element : $n_\mu d\Sigma$; $n_\mu n^\mu = \epsilon$

(95) ∴ The volume element of H.S. is $\sqrt{|h|}\, d^3 y = dV$

As Geodesics are moving with const value of y^a

∴ $d^3 y$ is same on each H.S.

∴ Only change comes from $\sqrt{|h|}$.

$$\delta V = \sqrt{h}$$

$$\frac{1}{\delta V}\frac{d}{d\tau}(\delta V) = \frac{1}{\sqrt{h}}\frac{d}{d\tau}\sqrt{|h|} = \frac{d \ln \sqrt{|h|}}{d\tau} = \frac{\partial \ln \sqrt{|h|}}{\partial y^i}\frac{dy^i}{d\tau}$$

As $\ln(\det M) = \mathrm{Tr}(\ln M)$

$$\partial_i h = h\, h^{ab}\, \partial_i h_{ab}$$

$$\partial_i \ln \sqrt{h} = \frac{h^{ab}}{2}\partial_i h_{ab}$$

$$\therefore \frac{h^{ab}}{2}\frac{\partial h_{ab}}{\partial y^i}\frac{dy^i}{d\tau} = \frac{h^{ab}}{2}\frac{d}{d\tau}(h_{ab})$$

96

$$\frac{dh_{ab}}{d\tau} = u^r \nabla_r h_{ab} = U^r \nabla_r (g_{\alpha\beta} e^\alpha_a e^\beta_b)$$

$$= u^r \cancel{(\nabla_r g_{\alpha\beta}) e^\alpha_a e^\beta_b} + u^r g_{\alpha\beta} (\nabla_r e^\alpha_a) e^\beta_b +$$

$$u^r g_{\alpha\beta} e^\alpha_a \nabla_r e^\beta_b$$

$$\mathcal{L}_u e^\alpha_a = u^\beta \nabla_\beta e^\alpha_a - e^\beta_a \nabla_\beta u^\alpha = 0$$

$$= g_{\alpha\beta} (u^r \nabla_r e^\alpha_a) e^\beta_b + g_{\alpha\beta} e^\alpha_a (u^r \nabla_r e^\beta_b)$$

$$= g_{\alpha\beta} (\nabla_r u^\alpha) e^r_a e^\beta_b + g_{\alpha\beta} e^\alpha_a e^r_b \nabla_r u^\beta$$

$$= (\nabla_r u_\beta) e^r_a e^\beta_b + e^\alpha_a e^r_b \nabla_r u_\alpha$$

$$= \nabla_\beta u_r \, e^\beta_a e^r_b + e^\beta_a e^r_b \nabla_r u_\beta$$

$$= (\nabla_\beta U_r + \nabla_r U_\beta) e^\beta_a e^r_b$$

$$= (B_{r\beta} + B_{\beta r}) e^\beta_a e^r_b$$

$$\frac{h^{ab}}{2} \frac{dh_{ab}}{d\tau} = \frac{B_{r\beta} + B_{\beta r}}{2} h^{ab} e^\beta_a e^r_b$$

$$= \frac{B_{r\beta} + B_{\beta r}}{2} h^{\beta r}$$

$$= B_{\beta r} h^{\beta r} = \theta$$

$$\frac{1}{\delta V} \frac{d(\delta V)}{d\tau}$$

in GNC $\theta = \frac{1}{\delta V} \frac{d\delta V}{d\tau}$ But as θ is scalar $\therefore$ in any coord. system

$$\theta = \frac{1}{\delta V} \frac{d(\delta V)}{d\tau}$$

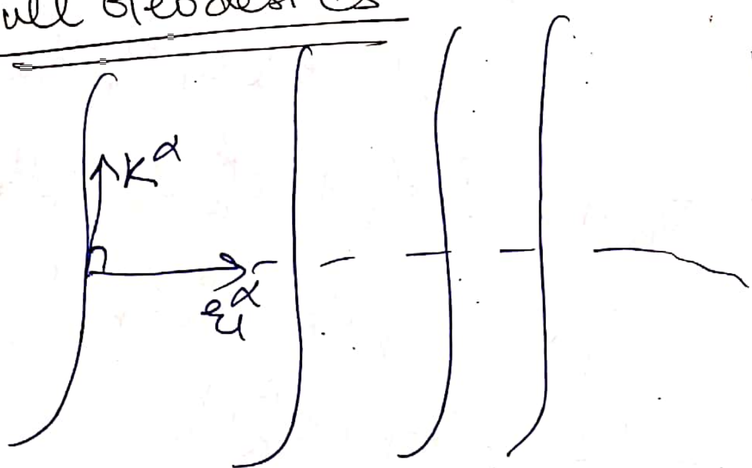
(97) In Cosmology

By Energy conservation $\nabla_\beta T^{\alpha\beta} = 0$
we get Transverse & Long. cosmological Eqn
& By Raychaudhuri Eqn we get one more Eqn
Now we have ρ, θ Eqn & 2 unknowns.

Def: $\delta V \propto a^3(t)$

$$\frac{\dot{a}}{a} \equiv H = \text{Hubble const.}$$

(98) Null Geodesics



Let the geodesic be Affinely parametrized.

Similar to timelike geodesic

$$k^\alpha \xi_\alpha = \text{const}.$$

Proof: $\frac{D(k^\alpha \xi_\alpha)}{d\tau} = k^\beta \nabla_\beta (k^\alpha \xi_\alpha)$

$$= k^\beta k^\alpha \nabla_\beta \xi_\alpha + (k^\beta \nabla_\beta k^\alpha)\xi_\alpha \qquad k^\alpha k_\alpha = 0 \text{ Everywhere on space}$$

$$= k^\beta \xi^\beta \nabla_\beta k_\alpha = \frac{\xi^\beta \nabla_\beta (k^\alpha k_\alpha)}{2} = 0$$

$$\therefore k^\alpha \xi_\alpha = \text{const}.$$

We can choose $k^\alpha \xi_\alpha = 0$ Because of the arbitrary parameterization. 101

(99) ∴ The following properties are held:

For affinely parametrized.

① $k^\beta \nabla_\beta k^\alpha = 0$ } Req. to be Affine. But ℓ was not Defined for null.

② $k^\alpha k_\alpha = 0$

③ $\mathcal{L}_{\xi} k^\alpha = -\mathcal{L}_k \xi^\alpha$ } Doesn't Require to be affine.

$\xi^\beta \nabla_\beta k^\alpha = k^\beta \nabla_\beta \xi^\alpha$

④ $k^\alpha \xi_\alpha = \text{const.}$ } Doesn't Req. to be Affine.

$k^\alpha \xi_\alpha = 0$

(100) ~~as~~ $\xi^\alpha k_\alpha = 0$ doesn't ensure $\xi \perp^r k$

as let $\xi^\alpha = c k^\alpha \Rightarrow c k^\alpha k_\alpha = 0$

∴ ξ^α can be in the direction of k^α

∴ $k^\alpha \xi_\alpha = 0$ fails to remove an eventual Component of ξ^α in the direction of k^α.

(101) ∴ Our first problem is to isolate purely transverse part of Deviation vector.

& ∴ to find metric which is transverse to ~~gab~~ k^α.

(102) ∴ As done for timelike vector in LIF

$$g_{\alpha\beta} \overset{*}{=} h_{\alpha\beta} + k_{\alpha} k_{\beta}$$

$$h_{\alpha\beta} \overset{*}{=} g_{\alpha\beta} - k_{\alpha} k_{\beta}$$

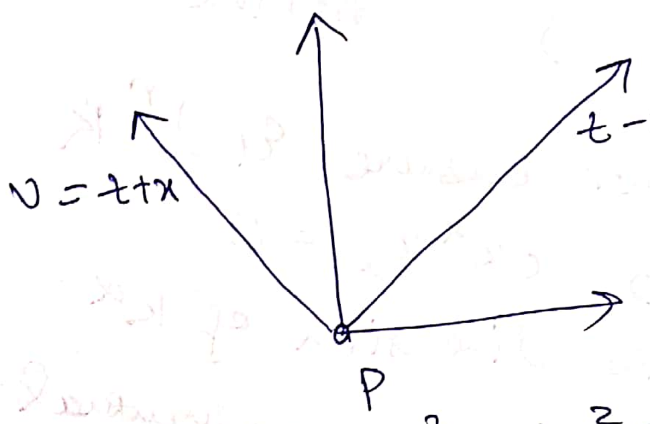
Tensorial Eqn $\Rightarrow$ $h_{\alpha\beta} = g_{\alpha\beta} - k_{\alpha} k_{\beta}$

$$h_{\alpha\beta} k^{\beta} = k^{\alpha} \neq 0$$

∴ $h_{\alpha\beta}$ is not transverse to k^{α}

∴ This Decomposition will not work.

(103) Go to LIF at Pt. P



$dt - dx = du$

$dt + dx = dv$

$2\,dt = du + dv$

$-2\,dx = du - dv$

$$ds^2 \overset{*}{=} dt^2 - dx^2 - dy^2 - dz^2$$

$$\overset{*}{=} \frac{1}{4}(2\,du\,dv + 2\,du\,dv) - dy^2 - dz^2$$

$$ds^2 \overset{*}{=} du\,dv - dy^2 - dz^2$$

Let k^{α} is tangent to $u =$ const surface

Transverse line element $\Rightarrow$ $d\tilde{s}^2 \overset{*}{=} -dy^2 - dz^2$ $\hookrightarrow$ 2D

(104) To isolate Transverse part of the metric we need to ~~isolate~~ introduce another Null vector field N_α 103

s.t. $N_\alpha k^\alpha \neq 0$

Impose condition $k^\alpha N_\alpha = +1$

let $k_\alpha \overset{*}{=} (1, -1, 0, 0)$

$N_\alpha \overset{*}{=} c(1, 1, 0, 0)$

s.t. $k_\alpha N^\alpha = 1$

$c + c = 1 \Rightarrow c = 1/2$

$\therefore N_\alpha = \partial_\alpha v = \frac{1}{2}(1, 1, 0, 0)$

(05) let $h_{\alpha\beta} = g_{\alpha\beta} - k_\alpha N_\beta - N_\alpha k_\beta$

$h_{\alpha\beta} k^\beta = k_\alpha - k_\alpha = 0$

$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha = 0$

$\boxed{\text{Earlier } h_{\alpha\beta} \text{ was not orth to } k^\alpha.}$

$k^\alpha h_{\alpha\beta} = N^\alpha h_{\alpha\beta} = 0$

$\therefore h_{\alpha\beta}$ is orth to k^α & N^α

$$h_{\alpha\beta} \overset{*}{=} \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$h_{\alpha\beta}$ is 2 Dimensional

(106) ∴ Given Null vector K^α

Select Auxilliary Null vector N_α

choose $K^\alpha N_\alpha = 1$

& $h_{\alpha\beta} = g_{\alpha\beta} - k_\alpha N_\beta - N_\alpha K_\beta$

It satisfies.

$$\left.\begin{array}{l} ① \; h_{\alpha\beta} K^\beta = h_{\alpha\beta} N^\beta = 0 \\ ② \; h^\alpha_\alpha = 2 \\ ③ \; h^\alpha_\mu h^\mu_\beta = h^\alpha_\beta \end{array}\right\}$$

$h^{\alpha\beta}$ Proj. Operator Orth. to K^α & N^α.

& $h^{\alpha\beta}$ is 2 Dim

(107) $$\left.\begin{array}{l} N^\alpha N_\alpha = 0 \\ K^\alpha N_\alpha = 1 \end{array}\right\}$$ 2 conditions to determine 4 components of N_α.

∴ 2 conditions & 4 variables

∴ Eqn Can't be solved for Unique Answer. θ Remains Inv. σ, ω X Remain Inv.

∴ $h_{\alpha\beta}$ is also Not Unique.

$\frac{d\theta}{d\lambda}, \sigma^{\alpha\beta}\sigma_{\beta\alpha}, \omega^{\alpha\beta}\omega_{\beta\alpha}$ Remains Inv.

⇒ ∴ Focusing Th. Remains Inv.

(108) $K^\beta \nabla_\beta \xi^\alpha_\varphi = \xi^\beta_\varphi \nabla_\beta \xi^\alpha_\varphi$

$\frac{D\xi^\alpha_\varphi}{d\lambda} = B^\alpha_\beta \xi^\beta_\varphi$

↳ Affinely Parameterized.

$B^\alpha_\beta K^{\alpha} = K^\beta B^\alpha_\beta = 0$

But $B^\alpha_\beta N^\alpha \neq 0$ ∴ B^α_β has Non Transverse component.

(109) $\tilde{\xi}_\varphi^\alpha = h^\alpha_\mu \xi_\varphi^\mu = \xi_\varphi^\mu \delta^\alpha_\mu - (k^\alpha N_\mu)\xi_\varphi^\mu - (N^\alpha k_\mu)\xi_\varphi^\mu$ 105

$= \xi_\varphi^\alpha - k^\alpha (N_\mu \xi_\varphi^\mu)$

has component along k^α

ξ_φ^α is $\perp^r$ to N^α

(110) $k^\beta \nabla_\beta \tilde{\xi}_\varphi^\alpha$ = Relative velocity of 2 Neighbouring Geodesic

$= k^\beta \nabla_\beta (\xi_\varphi^\alpha - k^\alpha (N_\mu \xi_\varphi^\mu))$

$= k^\beta \nabla_\beta \xi_\varphi^\alpha - k^\beta \nabla_\beta (k^\alpha (N_\mu \xi_\varphi^\mu))$

Why this didn't work?

$= \xi_\varphi^\beta \nabla_\beta k^\alpha - k^\beta \cancel{(\nabla_\beta k^\alpha)} N_\mu \xi_\varphi^\mu$

$- k^\beta k^\alpha \nabla_\beta (N_\mu \xi_\varphi^\mu)$

$= \xi_\varphi^\beta \nabla_\beta k^\alpha - k^\alpha k^\beta \nabla_\beta (N_\mu \xi_\varphi^\mu)$

$= \xi_\varphi^\beta \nabla_\beta k^\alpha - k^\alpha (k^\beta N_\mu \nabla_\beta \xi_\varphi^\mu) - k^\alpha (k^\beta \xi_\varphi^\mu \nabla_\beta N_\mu)$

$k^\beta \nabla_\beta \tilde{\xi}_\varphi^\alpha = k^\beta \nabla_\beta (h^\alpha_\mu \xi_\varphi^\mu)$.

$= (k^\beta \nabla_\beta h^\alpha_\mu) \xi_\varphi^\mu + k^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu$

$= k^\beta (\nabla_\beta (\delta^\alpha_\mu - k^\alpha N_\mu - N^\alpha k_\mu) \xi_\varphi^\mu) + k^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu$

$= -k^\beta \nabla_\beta (k^\alpha N_\mu) \xi_\varphi^\mu - \cancel{k^\beta (\nabla_\beta N^\alpha) k_\mu \xi_\varphi^\mu}$

$- k^\beta (\nabla_\beta k_\mu) N^\alpha \xi_\varphi^\mu + k^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu$

$= \cancel{-k^\beta (\nabla_\beta k^\alpha) N_\mu \xi_\varphi^\mu} - k^\beta k^\alpha (\nabla_\beta N_\mu) \xi_\varphi^\mu - \cancel{k^\beta (\nabla_\beta k_\mu) N^\alpha \xi_\varphi^\mu}$

$+ k^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu$

$$K^\beta \nabla_\beta \tilde{\xi}_\varphi^\alpha = -K^\beta K^\alpha (\nabla_\beta N_\mu) \xi_\varphi^\mu + K^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu$$ 106

$$= K^\beta h^\alpha_\mu \nabla_\beta \xi_\varphi^\mu - K^\alpha \left(K^\beta (\nabla_\beta N_\mu) \xi_\varphi^\mu\right)$$

$$= \xi_\varphi^\beta h^\alpha_\mu \nabla_\beta K^\mu - \underbrace{K^\alpha \left(K^\beta (\nabla_\beta N_\mu) \xi_\varphi^\mu\right)}_{\text{component along } K}$$

(111)

Removing this component To get Purely Transverse velocity

$$K^\beta \nabla_\beta \tilde{\xi}_\varphi^\alpha = h^\alpha_i \left(K^\beta \nabla_\beta \tilde{\xi}_\varphi^i\right)$$

$$= h^\alpha_i \left(\xi_\varphi^\beta h^i_\mu \nabla_\beta K^\mu - K^i \left(K^\beta (\nabla_\beta N_\mu) \xi_\varphi^\mu\right)\right)$$

$$= h^\alpha_\mu \left(\xi_\varphi^\beta \nabla_\beta K^\mu\right) - \cancel{(h^\alpha_i K^i)}\, K^\beta (\nabla_\beta N_\mu) \xi_\varphi^\mu$$

$$= h^\alpha_\mu \xi_\varphi^\beta \nabla_\beta K^\mu$$

$$= h^\alpha_\mu \left(\tilde{\xi}_\varphi^\beta - K^\beta (N_i \xi_\varphi^i)\right) \nabla_\beta K^\mu$$

$$= h^\alpha_\mu \tilde{\xi}_\varphi^\beta \nabla_\beta K^\mu$$

$$\tilde{\xi}_\varphi^\beta = h^\beta_i \xi_\varphi^i$$

$$= h^\beta_\mu h^\mu_i \xi_\varphi^i$$

$$\tilde{\xi}_\varphi^\beta = h^\beta_\mu \tilde{\xi}_\varphi^\mu$$

$$K^\beta \nabla_\beta \xi_\varphi^\alpha = h^\alpha_\mu h^\beta_i \tilde{\xi}_\varphi^i \nabla_\beta K^\mu$$

$$= h^\alpha_\mu h^\beta_i B^\mu_\beta \tilde{\xi}_\varphi^i = \tilde{B}^\alpha_i \tilde{\xi}_\varphi^i$$

(112) Define

Purly Transverse Part $B_{\alpha\beta} = \nabla_\beta k_\alpha \equiv \tilde{B}_{\alpha\beta}$ how?

$$\tilde{B}_{\alpha\beta} = h^{\mu}_{\alpha} h^{\nu}_{\beta} B_{\mu\nu} \Longleftrightarrow \tilde{B}^{\alpha}_{i} = h^{\alpha}_{r} h^{\beta}_{i} B^{r}_{\beta}$$

(113) $\tilde{B}_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{2} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$

Similarly as in Timelike Geodesic Case.

(114) $h^{\alpha\beta}\sigma_{\alpha\beta} = 0$ ($h^{\alpha\beta}$: 2D), $h^{\alpha\beta}\omega_{\alpha\beta} = 0$ $\Big\}$ $\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta}$

(115) $\tilde{B}_{\alpha\beta} k^{\alpha} = h^{\mu}_{\alpha} h^{\nu}_{\beta} B_{\mu\nu} k^{\alpha}$

$$= h^{\nu}_{\beta}(h^{\mu}_{\alpha} k^{\alpha}) B_{\mu\nu} = 0$$

as $\boxed{h_{\alpha\beta} k^{\beta} = 0 \quad h_{\alpha\beta} N^{\beta} = 0}$

$$\tilde{B}_{\alpha\beta} N^{\alpha} = h^{\mu}_{\alpha} h^{\nu}_{\beta} B_{\mu\nu} N^{\alpha}$$

$$= h^{\nu}_{\beta}(h^{\mu}_{\alpha} N^{\alpha}) B_{\mu\nu} = 0$$

$\therefore$ $\boxed{B_{\alpha\beta} k^{\alpha} = 0 \quad \text{But} \quad \tilde{B}_{\alpha\beta} N^{\alpha} = \tilde{B}_{\alpha\beta} k^{\alpha} = 0}$

(116) $h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta}$

Due to (115)

(117) $\tilde{B}^{\alpha}_{i} = B^{\alpha}_{i} - k^{\alpha}(B^{\gamma}_{i}N_{\gamma}) - (N^{\beta}B_{\beta}{}^{\alpha})k_i + k^{\alpha}k_i(N_{\gamma}N^{\beta}B_{\beta}{}^{\gamma})$

kills N components

as $B^{\alpha}_{i}k^{i} = 0$

B^{α}_{i} has N components.

(118) By (114)

$$\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}B_{\alpha\beta} = \nabla_{\beta}k^{\beta}$$

(119) $\theta = \nabla_{\beta}k^{\beta}$ doesn't depend on the choice of Auxilary Null vector N^{α}

$\therefore$ θ is Unique.

(120) Shear & Rotation Depends on choice of Null Direction.

Why?

(121) As $\kappa^\alpha \xi_\alpha = 0$, $k^\beta \nabla_\beta k^\alpha = 0$, $\mathcal{L}_\xi K^\alpha = 0$

Can $N^\alpha \xi_\alpha = 0$, $N^\beta \nabla_\beta N^\alpha = 0$, $\mathcal{L}_\xi N^\alpha = 0$?

Frobenius Theorem

(122) As Earlier for Congruence (T/N/S) we have proved congruence is H.S.O. iff $(\nabla_{[\beta} k_\alpha) k_{\gamma]} = 0$

Now is H.S.O. iff $\omega_{\alpha\beta} = 0$.

Th. Null Geod. is H.S.O. iff

Proof: $B_{[\alpha\beta]} k_\gamma + B_{[\gamma\alpha]} k_\beta + B_{[\beta\gamma]} k_\alpha = 0$

N^γ

$$B_{[\alpha\beta]} = B_{[\gamma\alpha]} k_\beta N^\gamma + B_{[\beta\gamma]} k_\alpha N^\gamma$$

From (117)

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} - k_\alpha (B^\gamma_\beta N_\gamma) - (N^\beta B_{\alpha\beta}) k_\beta + k_\alpha k_\beta (N_\gamma N^\delta B^\gamma_\delta)$$

$(N_\gamma B^\gamma_\alpha) k_\beta$

$$\Rightarrow \tilde{B}_{[\alpha\beta]} = B_{[\alpha\beta]} - k_{[\alpha} (B^\gamma_{\beta]} N_\gamma) - (N^\gamma B_{\gamma[\alpha}) k_{\beta]}$$

But $B_{[\alpha\beta]} = B_{\gamma[\alpha} k_{\beta]} N^{\gamma} + k_{[\alpha} B_{\beta]\gamma} N^{\gamma}$

$\therefore \tilde{B}_{[\alpha\beta]} = 0 = \omega_{\alpha\beta}$

(123) **Theorem**
If A null vector field is H.S.O then it satisfies Geodesic eqn.

~~Visualization: Every null vector field is not affinely parameterized~~

$K_\alpha = \partial_\alpha \phi$

$$\boxed{\begin{array}{l} K^\beta \nabla_\beta K_\alpha = \partial_\alpha (\partial_\beta \phi \partial^\beta \phi)/2 \\ f = \partial_\beta \phi \partial^\beta \phi = 0 \text{ on } \Sigma \\ \partial_\alpha f \text{ is } \perp^r \text{ to } \Sigma \\ \therefore \partial_\alpha f \propto K_\alpha \\ \therefore K^\beta \nabla_\beta K_\alpha = \kappa K_\alpha \end{array}}$$

Proof: $K_\alpha = A \partial_\alpha \phi$

$$K^\beta \nabla_\beta K_\alpha = K^\beta \partial_\beta (A \partial_\alpha \phi)$$
$$= K^\beta \partial_\beta A \partial_\alpha \phi + K^\beta A \partial_\beta \partial_\alpha \phi$$
$$= A \partial^\beta \phi \, \partial_\beta A \partial_\alpha \phi + (\partial^\beta \phi) A \partial_\beta \partial_\alpha \phi$$

$$\partial^\beta \phi \partial_\beta \partial_\alpha \phi = \frac{\partial_\alpha (\partial_\beta \phi \partial^\beta \phi)}{2} = 0$$

$\therefore$ Every Null ~~vector field~~ Curve on H.S is Geodesic ~~Only for Non affine Parameter~~

$$K^\beta \nabla_\beta K_\alpha = (\partial^\beta \phi \, \partial_\beta A) K_\alpha$$

If there is stack of Null H.S

$\partial_\alpha (\partial^\beta \phi \partial_\beta \phi) = \partial_\alpha f = 0$

$\therefore \kappa = 0$

In this case λ is affine Parameter

It is a Geodesic eqn.

(124) $\therefore$ Null vector field which is H.S.O is a Geodesic.

(125) Affine Parametrization is recovered when $(\partial_\alpha A) K^\alpha = 0$ i.e. A does not vary along the geodesic.

($\partial_\alpha A$) from Timelike Case. !!!

(126) Th. Every Null Vector field is H.S.O

$n_\alpha = A\,\partial_\alpha \phi$

$n_\alpha n^\alpha = 0$ (Null vector)

$\therefore$ n^α lie on the H.S

$\therefore$ $n^\alpha \propto k^\alpha$

$\therefore$ Let $k_\alpha = \partial_\alpha \phi \Rightarrow$ H.S.O.

Summary
(1) whenever we are given null H.S, we are given null curves;
(2) null curves on H.S are geodesic;
(3) null curves on H.S are H.S.O.

(127) Raychaudhuri Eqn for Null Geodesic

See L-830

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta$$

This Eqn is invariant under change of Auxilary Null vector N^α.

(128) Focusing Theorem

for Null Case

Strong Engy Condition $\Longleftrightarrow$ Weak Null Energy Condn

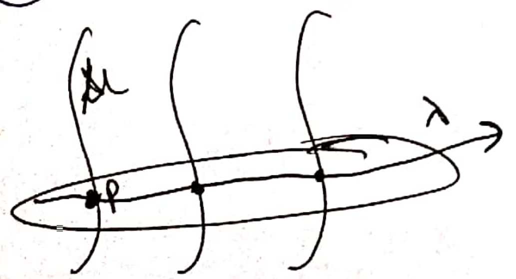
(129) $\theta(\lambda) \to -\infty$

within affine parameter $\lambda \leq \frac{2}{|\theta_0|}$

(130) For Example:

Proof ?

(131) $\theta = \frac{1}{\delta A} \frac{d}{dx}(\delta A)$



A

Ch-3
Hyper Surfaces

① Σ eqn ① $\phi(x^\alpha) = 0$

② Parametric

$x^\alpha = x^\alpha(y^a)$

② Normal $n_\alpha \propto \partial_\alpha \phi$

for Non Null $\quad n_\alpha n^\alpha \equiv \epsilon \equiv \begin{cases} 1 & \Sigma \text{ spacelike} \\ -1 & \Sigma \text{ timelike} \end{cases}$

Normalized

Convention:& n^α point in the direction of increasing ϕ

$\therefore n^\alpha \partial_\alpha \phi > 0$??

③ From these 2 we can get that

$$n_\alpha = \frac{\epsilon \, \partial_\alpha \phi}{|g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi|^{1/2}}$$

④ This will not work for Null case

as $\partial_\mu \phi \partial^\mu \phi = 0$

$\therefore n_\alpha \longrightarrow \infty$

⑤ Hence normal can't be Normalized in Null Case

$\therefore$ let $k_\alpha = \partial_\alpha \phi$

when $\phi \uparrow$ in future k_α should be future directed

⑥ As $k^\alpha k_\alpha = 0 \quad \therefore k^\alpha$ is Tangent to Σ