A study on black hole quasi normal modes

Report on the Vritika Internship Program
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Submitted by
Ashley Chraya

Under the Guidance of
Dr. Srijit Bhattacharjee

Organized at Indian Institute of Information Technology, Allahabad

Declaration

I declare that this written report represents my ideas in my own words and where others ideas or words have been included, I have adequately cited and referenced the original sources. This project report is an extended literature review with my own comments all over the report.

I also declare that I have adhered to all principles of academic honesty and integrity. I also declare that I have not submitted this summer report at any other institution.

Date: 26/07/2021

Ashley Chraya

Abstract

In this report, we discuss two issues regarding the quasinormal modes. Firstly, we explain how the lyapunov exponents magically appear in the the formula of oscillation frequency through WKB method, giving us a trick to measure these frequency through particle dynamics around photon ring and without perturbing the field around the black hole spacetime. We further find that this analogy between particle dynamics and scalar dynamics is just a coincidence and has no deeper meaning as when we probe into higher order WKB approximation, we find no such analogy to exist. Another important issue discussed is the role of the boundary condition in case of black hole system. We explicitly show that perturbed black hole systems are intrinsically dissipative due to the boundary conditions adopted for such system.

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List of Abbreviations

BH	Black holes
GR	General Relativity
LIGO	Laser Interferometer Gravitational-Wave Observatory
QNM	Quasinormal mode
SR	Special Relativity

Chapter 1

Introduction

1.1 Motivation

Characterstics modes of vibrations are ubiqutous in nature around us, and black holes are no exception to this. The characterstic oscillations of black holes, called quasi-normal modes are produced when the background black hole spacetime is perturbed. The situation here is quite similar to the system we are familiar with where one usually perturbs the system from equilibrium in Newtonian mechanics and quantum mechanics, and studies its properties. For example, when the bell is hit with a hammer, and disturbed from the equilibrium point, it starts ringing and then the oscillations fades away after some time. Similarly, black holes are disturbed by any test field, for example two black holes inspiraling towards each other until they merge into one single black hole, and consequently the resulting black hole starts producing characterstic oscillations by which the properties of black holes are studied.

Infact, these binary black holes coalescence can be divided into three phases: a) Inspiral phase b) Merger phase c) Ringdown. In the inspiral phase two blackholes orbit around each other with shrinking orbital radius due to emission of gravitational waves. In this phase, the black holes are separated widely enough to be treated as point particles. As the orbits radius decreases, angular frequency increases, and the emission of gravitational waves increases. When the holes get so close together (innermost stable circular orbit (ISCO)) that they can no longer be approximated as point particles, they enter the merger phase. In the merger phase, two black holes collide and the gravitational wave emission is at peak in this phase. In this phase single, highly distorted black hole is formed at the end. After the merger, this remnant black hole will produce these damping oscillations called as quasinormal modes.Hence, black hole binaries are the rich source of gravitational waves for detection by LIGO.

One important property which is studied in black hole mechanics is the stability of the system i.e. what would happen to black holes when they are perturbed from their initial state. The question we try to answer in this analysis is equivalent to how the car parked on the top of the hill would behave after it is disturbed. The criterion in case of black hole for stability is when there damping of the gravitational waves i.e. negative imaginary values of the characterstic modes.

The quasi-normal frequencies are also important in gravitational wave astronomy, as these are an important source of gravitational waves emitted at discrete frequencies by a perturbed BH. QNMs have discrete, complex frequencies, where imaginary part represents the decay timescale of the perturbation, and real part determine the oscillation frequency [1]. Gravitational wave detectors observe these frequencies in the last ringdown phase of the binary mergers. By analyzing, the ringdown behavior, we can determine different chacterstics of the black hole like their mass, charge, and angular momentum as Quasinormal frequencies serve as a fingerprint for oscillating black holes. By numerical calculations of QNMs through different methods we can get the values of frequency at which LIGO should operate to check for the the signals of gravitational waves.

Detection of these frequencies also confirm the existence of the BHs. From 1 April to 1 October 2019, the upgraded LIGO and Virgo interferometers detected 39 new gravitational wave events from massive collisions between neutron stars or black holes. This has given us the wide range of black holes that not only had never been detected before, but can reveal uncharted territories of the evolution and afterlives of binary stars.

1.2 Scope of this report in a nutshell

In chapter 2, we give a brief review of particle dynamics around Schwarszchild black hole and lyapunov exponent in case of null circular orbits. In chapter 3, we give a brief review of scalar field dynamics around spherically symmetric and static black hole. Further, we provide arguments of why the boundary conditions are important in every physical system and what are its consequences in case of black hole. In chapter 4, we provide basic introduction of and numerically calculate oscillation frequency by WKB method (approximate) and Leaver or continued fraction method (Numerically exact) for Schwarszchild and RN black hole.

Chapter 2

Particle dynamics around Black hole

2.1 Overview

The main motivation of this chapter is to introduce the concept of lyapunov exponent for null circular orbits which will play an important role to make a point regarding the analogy between particle dynamics at photon ring and the scalar dynamics.

2.2 Static and Spherical Symmetric metric

The spherically symmetric and static metric describing spherical star can be written in the form of:

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.1}$$

This metric can be reduced to Schwarschild metric in case of $f(r) = h(r) = 1 - 2M/r$, and to Reissner-Nordstr metric in case of $f(r) = h(r) = 1 - 2M/r + Q^2/r^2$. For the above general metric, lagrangian takes the form:

$$\mathscr{L} = g_{\mu\mu}\dot{x}^\mu\dot{x}^\nu = -ft^2 + h^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{2.2}$$

One can easily see that the above lagrangian does not depend on t and ϕ. Therefore, the corresponding killing vectors inner product with the tangent vector along the affine geodesic remains constant along the geodesic motion, and hence yielding two conserved quantities E and L:

$$k_t^\mu U_\mu = -(1 - \frac{2M}{r})\frac{dt}{d\lambda} = f\dot{t} = \tilde{E} \tag{2.3}$$

$$k_\phi^\mu U_\mu = r^2\sin^2\theta\frac{d\phi}{d\lambda} = r^2\sin^2\theta\dot{\phi} = \tilde{L} \tag{2.4}$$

As the problem we are dealing with is central force problem, the orbit will be confined to one fixed plane let say equatorial plane $\theta = \pi/2$ at all times. Hence,

$$f\dot{t} = \tilde{E} \tag{2.5}$$

$$r^2\dot{\phi} = \tilde{L} \tag{2.6}$$

Assuming affinely paramterized geodesic the inner product of tangent vector with itself is constant (1 for the massive particle and 0 for the massless particle). This implies

$$\frac{E^2}{f} - \frac{\dot{r}^2}{h} - \frac{L^2}{r^2} = \delta \tag{2.7}$$

where $\delta = 0$ for null geodesics and $\delta = 1$ for timelike geodesics. From the above equation, one can obtain familiar radial equation used to study effective potential

$$\dot{r}^2 = V_r \tag{2.8}$$

$$V_r \equiv h\left(\frac{E^2}{f} - \frac{L^2}{r^2} - \delta\right) \tag{2.9}$$

Angular frequency of the orbit is found out to be

$$\Omega_c \equiv \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{fL}{r^2E} = \sqrt{\frac{f}{r^2}} = \sqrt{\frac{f'}{2r}} \tag{2.10}$$

2.2.1 Lyapunov Exponents for Circular Orbits

Lyapunov exponents are the measure of the rate at which the trajectories diverge. We can also define lyapunov exponent as the inverse of the instability timescale $\lambda_0 \equiv 1/T_{\lambda_0}$. Any dynamical system can be written in the form of

$$\frac{dX_i}{dt} = H_i\left(X_j\right) \tag{2.11}$$

By linearizing the system about any certain orbit we get,

$$\frac{d\delta X_i(t)}{dt} = K_{ij}(t)\delta X_j(t) \tag{2.12}$$

The solution of the above equation can be assumed as

$$\delta X_i(t) = L_{ij}(t)\delta X_j(0) \tag{2.13}$$

where we can easily find the conditions which must be followed by L_{ij} at all times for the solution to be held true. The eigenvalues of this matrix L_{ij} are known as lyapunov exponents, and the principal lyapunov exponent is defined as the largest of these eigenvalues, such that

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{L_{jj}(t)}{L_{jj}(0)}\right) \tag{2.14}$$

Now, applying it for the circular orbits in static, and spherically symmetric spacetimes, we find the linearized equation about the circular orbit as,

$$\dot{\delta} r = [\frac{1}{g_{rr}}]_{r=r_c} \delta p_r \tag{2.15}$$

$$\delta \dot{p}_r = [\frac{d}{dr}\left(\frac{\partial \mathscr{L}}{\partial r}\right)]_{r=r_c} \delta r \tag{2.16}$$

where dots represents derivatives w.r.t. λ which is an affine parameter. Now converting to Schwarzschild time t, we get

$$K_{ij} = \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix} \tag{2.17}$$

where

$$K_1 = t^{-1} \frac{d}{dr}\left(\frac{\partial \mathscr{L}}{\partial r}\right) \tag{2.18}$$

$$K_2 = (\dot{t} g_{rr})^{-1} \tag{2.19}$$

From the definition (2.14), principal lyapunov exponent comes out to be [1]

$$\lambda_0 = \pm\sqrt{K_1 K_2} \tag{2.20}$$

From the effective potential (2.8) for the particle in static and spherical symmetric spacetime, we can find the expression for lyapunov exponent in terms of potential.

$$\lambda_0 = \sqrt{\frac{V_r''}{2\dot{t}^2}} \tag{2.21}$$

We can conclude that real values of lyapunov exponents correspond to unstable orbits.

Null Geodesics

Taking double derivative of equation (2.8), and evaluating at $r = r_c$ of the circular null geodesic, we obtain

$$V_r''(r_c) = \frac{h_c}{f_c}\frac{L^2}{r_c^4}\left[2f_c - r_c^2 f_c^m\right] \tag{2.22}$$

By using torotise coordinate, and angular coordinate (2.10, we can convert the lyapunov exponent expression (2.21) in the form,

$$\lambda_0 = -\frac{1}{\sqrt{2}}\sqrt{\frac{r_c^2 f_c}{L^2} V_r^n(r_c)} = \frac{1}{\sqrt{2}}\sqrt{-\frac{r_c^2}{f_c}\left(\frac{d^2}{dr_*^2}\frac{f}{r^2}\right)_{r=r_c}} \tag{2.23}$$

[1]The proof of the (2.20) can be found in appendix A.1.

2.3 Schwarzschild metric

Till now, we were dealing with the general spherical symmetric metric, but now let us assume the schwarschild metric. The effective potential in this case would reduce to

$$V_r = E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L}{r^2} + \delta\right) \tag{2.24}$$

which can be further interpreted as

$$\dot{r}^2 = E^2 - 2V_\delta \tag{2.25}$$

where,

$$V_{part}(r) = 2\mathscr{V}_{\delta_{=1}}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{2.26}$$

$$V_{phot}(r) = \frac{2\mathscr{V}_{\delta_{=0}(r)}}{L^2} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \tag{2.27}$$

Condition for circular geodesic ($\dot{r} = \ddot{r} = 0$) reduces to

$$L^2(r - 3M) = \delta M r^2 \tag{2.28}$$

From this condition, we can make two important remarks: (i) Circular orbits can only exist for $r \geq 3M$ (ii) Null Circular orbits exist only at $r = 3M$.
We can also analyse the stability of the circular orbit by checking the sign of $V_r^{''}$ (sign of $V_r^{''}$ is opposite of V''_{part} and V''_{phot}, hence, stability conditions would reverse their sign): (i) Stable Orbit: $V_r'' < 0$, Unstable Orbit: $V_r'' > 0$
$V_r^{''}$ reduces to the form

$$(V_r)'' = -\frac{2\delta_1 M(r - 6M)}{r^3(r - 3M)} \tag{2.29}$$

Another two important remarks about the circular orbits can be made by analysing the above equation.(i) Circular orbits with $r \geq 6M$ are stable. (ii) Circular orbits with $3M \leq r < 6M$ are unstable. Alternatively, we can analyse the graph of effective potential for the case of timelike geodesic, and conclude that the inflection point of the potential is at radius 6GM, above which stable circular orbits can be found.

2.4 Bibliography Notes

This whole chapter was greatly influenced by chapter 2 from [2] and [3].

Chapter 3

Scalar Field Dynamics around Black hole

3.1 Overview

The main motivation of this chapter is to illustrate the role of boundary conditions in any physical system and in particular what are the consequences of the usual boundary conditions in case of black hole.

3.2 Static and Spherical Symmetric metric

Consider the perturbation of the spherical symmetric and static spacetime by a massive scalar field. Klein Gordan equation in this background can be written as :

$$(\Box - \mu^2)\phi = 0 \tag{3.1}$$

Decomposing the scalar field into spherical harmonics $Y_{\ell m}(\theta,\phi) = P_{\ell m}(\theta)e^{im\phi}$ (orthonormal set of functions), time harmonics and radial part (which is indeed due to time-independence and the spherical symmetry of the metric) as

$$\Psi(t,r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\Psi_{\ell m}^{s=0}(r)}{r} P_{em}(\theta)e^{-i\omega t}e^{im\phi} \tag{3.2}$$

where $P_{lm}(\theta)$ is the associated legendre function satisfying legendre equation. Inserting the above equation in the equation (3.1) and using legendre equation, we get the radial wave eqauation for a spherical symmetric and static background:

$$fh\frac{d^2\Psi_l^{s=0}}{dr^2} + \frac{1}{2}(fh)\frac{d\Psi_l^{s=0}}{dr} + \left[\omega^2 - \left(\mu^2 f + \frac{\ell(\ell+1)}{r^2}f + \frac{(fh)'}{2r}\right)\right]\Psi_l^{s=0} = 0 \tag{3.3}$$

We can reduce the above equation into Schrodinger type (removal of first order derivative)[1]by using tortoise coordinate $r_\star$ defined as

$$\frac{dr}{dr_*} \equiv (fh)^{1/2} \tag{3.4}$$

[1]Removing first order derivative to solve a second order differential equation is a common method found in textbooks. Therefore, the aim of using tortotise coordinates is exactly this i.e. to remove the first order derivative.

to obtain the following radial equation:

$$\frac{d^2\Psi_l^{s=0}}{dr_*^2} + \left[\omega^2 - V_0(\mu)\right]\Psi_l^{s=0} = 0 \tag{3.5}$$

where effective potential reads as

$$V_0(\mu) \equiv f\mu^2 + f\frac{\ell(\ell+1)}{r^2} + \frac{(fh)^r}{2r} \tag{3.6}$$

We can see the above differential equation doesn't depend on m as it shouldn't be because we are dealing with spherically symmetric spacetime.
The range in terms of tortoise coordinate is from $r_\star \to -\infty$ (corresponding to $r \to 2M$) to $r_\star \to \infty$ (corresponding to $r \to \infty$) which easily be seen by integrating the equation (3.4) and using common choice of integrating constant in literature :

$$r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right) \tag{3.7}$$

When the effective potential (3.6) is written in terms of $r_\star$, we can make three important remarks about the asymptotic behavior of potential
(i) It tends to zero (exponential decay $\sim \exp^(r_\star/M)$)as $r_\star/M \to -\infty$
(ii) It tends to μ^2 (power law tail$\sim \frac{l(l+1)}{r^2}$) as $r_\star/M \to \infty$.
(iii)Potential tends to zero in both asymptotic regimes unless $\mu \neq 0$

Our aim is now to solve for ω in the radial equation. Viewing it as eigenvalue problem, the problem boils down to finding eigen values of $\mathscr{L}$ operator where

$$\mathscr{L} = \frac{\mathrm{d}^2}{\mathrm{d}r_*^2} - V \tag{3.8}$$

3.3 Role of Boundary Condition

Taking a step back, let us first analyze the normal modes in physical system that we are familiar with, e.g. a string of length L that is fixed at both ends. The oscillations on the string can be described by the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{T}{\rho}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0 \tag{3.9}$$

where ρ and T describes the density and tension in the string, reswpectively. The general solution of the above equation is

$$y(t,x) = \sum_{n=-\infty}^{\infty} c_n \exp(\mathrm{i}\omega_n t)\,\Phi(x), \tag{3.10}$$

where ω_n is the normal mode spectrum and constants c_n describes the excitation of the given mode. In order to determine specific solution for $y(x)$, one has to provide boundary condition, e.g. Dirichlet type $[y(t,x=0), y(t,x=L)]$ and initial conditions $[y(t=0,x), y^{(t)}(t=0,x)]$. For a standing wave on an infinite length string, one gets the wave solution

$$y(x,t) = 2y_{\text{max}} \sin\left(\frac{2\pi x}{\lambda}\right) \cos(\omega t) \tag{3.11}$$

As there are no boundary conditions in this case, ω can take any value i.e. spectrum is continuous and infinite.
Similarly, if we take the specific case of standing wave on a finite length string, then there would be two boundary conditions at the strings end.

$$y(0,t) = 0 \tag{3.12}$$

$$y(L,t) = 2y_{\text{max}} \sin\left(\frac{2\pi L}{\lambda}\right) \cos(\omega t) = 0 \tag{3.13}$$

The latter boundary condition leads to the restriction on the frequency of the standing waves

$$f = \frac{nv}{2L} \tag{3.14}$$

where v is the speed of the wave along the string.
By the above examples one can identify the crucial difference between ω_n and c_n. The frequency spectrum are determined by the boundary conditions and are independent of the initial conditions. We cna also see from above examples that there are many ways to excite the system (depending on the initial conditions), but the set of possible modes doesn't depend on this. It may happen that some modes are not excited for some initial conditions, but it is not possible to excite the mode which is not the part of the mode spectrum. Therefore, to find any fundamental property of the system for e.g. density or tension in the string, the knowledge of the modes ω_n is essential. Therfore, we can reach the conclusion that boundary conditions are important to know fundamental properties of the system.
Black hole systems are no exception to this, and therefore we need to specify boundary conditions here[4], in order to solve the differential equation(3.5). As potential tends to zero in both asymptotic regimes in case of massless scalar field, therefore, solution of the radial equation (3.5) near horizon and at spatial infinity can be written as $\Psi \sim e^{-i\omega(t \pm r_+)}$. Classically nothing should leave the horizon, therefore, only ingoing wave should be present,

$$\Psi \sim e^{-i\omega(t+r_+)} \tag{3.15}$$

By discarding the incoming waves from infinity and considering the massless scalar field, the radial equation (3.5) solution in at spatial infinity range is

$$\Psi \sim e^{-i\omega(t-r_+)} \tag{3.16}$$

The nature of ω depends on the boundary conditions as was in the simple case of vibrations of string. In the case of black holes, we find that the frequency is a complex variable. The imaginary part of the frequency is non zero due to this boundary condition. The consequence of this is that the eigenfunctions becomes damped in nature, and do not form the complete set. Therefore, the perturbed black hole spacetimes are intrinsically disspative due to the boundary conditions. Now, we are left with to show explicitly how the boundary conditions make the modes complex and why imaginary part of it is associated with the decay timescale of the perturbation. The latter is easy to show. Let $\omega = \omega' + i\omega//$

$$\Psi = Re(e^{i\omega t}) = e^{-\omega'' t}\cos(\omega' t) \tag{3.17}$$

where $e^{-\omega'' t}$ is the decaying factor, and hence imaginary part is associated with decay timescale, and real part associated with the oscillation frequency.

3.4 Consequences of Boundary Condition

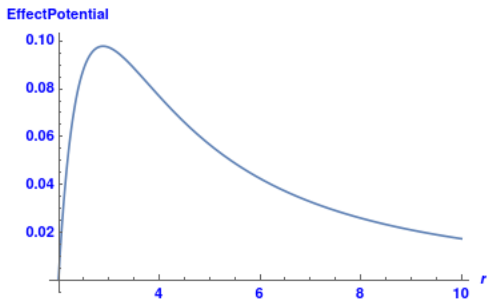

Figure 3.1: Graph of effective potential for the case of scalar field at $\ell = 1$

As from the figure we can observe that thre is a potential barrier against an incoming wave. To analyse how the incoming waves are affected, consider the incoming wave of the form $Phi = \exp(-i\omega x)$. If $R(\omega)$ and $T(\omega)$ are the reflection and transmission coefficients respectively, the reflected and transmission terms are $R(\omega)\exp(i\omega x)$ and $T(\omega)\exp(-i\omega x)$. Therefore due to superposition we can write the wave form near spatial infinity as $Phi = \exp(-i\omega x) + R(\omega)\exp(i\omega x)$, and near horizon as $\Phi = T(\omega)\exp(-i\omega x)$. The radial differential equation has a constant Wronskian therefore we can write

$$W = [\psi, \psi^*] = \frac{d\psi}{dx}\psi^* - \frac{d\psi^*}{dx}\psi = -2i\omega|T(\omega)|^2$$

as $x \to -\infty$, and we can write

$$W = 2i\omega\left(|R(\omega)|^2 - 1\right)$$

as $x \to \infty$. As wronskian is constant therefore, we get

$$|R(\omega)|^2 + |T(\omega)|^2 = 1 \tag{3.18}$$

which is the condition of conservation of energy.
Now, applying boundary condition that only outgoing waves exist at spatial infinity. By applying these boundary conditions we get

$$|R(\omega)|^2 = -|T(\omega)|^2 \tag{3.19}$$

The above relation has two consequences:
1) By above equation, we conclude that if the usual boundary conditions are followed then the amplitude of the reflected and transmitted waves is equal.

2)We can also conclude from the above equation that frequency ω is a complex number. Hence, the oscillations from BH are damped/growing depending on sign of imaginary part of the oscillation. Equation (3.19) doesn't hold in Kerr metric spacetime which is shown in appendix A.3.

3.5 Bibliography Notes

Section 3.2 is influenced from [2] and rest of the sections were influenced by [4] and [5].

Chapter 4

Analysis of Quasinormal modes

4.1 Overview

Another way to show explicitly how the boundary conditions make the modes complex, we have to find the eigen values of the radial differential equation. We will adopt two methods here to solve the differential equation
(i) Numerically exact method - Leaver's method or continued fraction method.
(ii) Approximate method - Wentzel–Kramers–Brillouin (WKB) approximation method.
We will review both the methods in this chapter.

4.2 Master Equation

As earlier, we separated the angular dependence for the scalar field, the angular dependence can also be removed for spin-one and spin-two case via vector and tensor spherical harmonics. As a result, we get the master equation which takes the similar form as the radial equation in case of scalar field,

$$\frac{d^2\Psi_f^*}{dr_*^2} + \left[\omega^2 - V_*\right]\Psi_f^* = 0 \tag{4.1}$$

where

$$V_s = f\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M\left(1-s^2\right)}{r^3}\right) \tag{4.2}$$

$$= \left(1 - \frac{2GM}{r}\right)\left(\frac{\Lambda}{r^2} + \frac{2\beta GM}{r^3}\right) \tag{4.3}$$

Assuming $Q(r) = \omega^2 - V_s$, differentiating $Q(r_\star)$ with respect to $r_\star$ coordinate, we obtain

$$\frac{dQ}{dr_*} = \left(1 - \frac{2GM}{r}\right)\left[\frac{2\Lambda}{r^4}(r - 3GM) + \frac{2\beta GM}{r^5}(3r - 8GM)\right]$$

For an extremum, $Q'(r_\star) = 0$, we get the condition,

$$\frac{2\Lambda}{r^4}(r - 3GM) + \frac{2\beta GM}{r^5}(3r - 8GM) = 0$$

From the above condition we get the position where the extrema occurs at,

$$r_0 = \frac{3GM(\Lambda-\beta) \pm \sqrt{9G^2M^2(\beta-\Lambda)^2 + 4\cdot\Lambda\cdot 8\beta G^2M^2}}{2\Lambda} \tag{4.4}$$

Taking the positive value and taking eikonal limit (large Λ limit), the radius approximates to 3GM.

$$r_0 \approx 3GM, \quad \text{as} \quad \Lambda \to \infty \tag{4.5}$$

Even for small l's the location of the peak is very close to the photon orbit.[6]

4.3 Leaver's Method

Now, we want to illustrate the theory developed by Leaver, commonly known as continued fraction method, to numerically search for the roots of the radial equation(3.5). Taking 2M=1, equation(3.5) can be written as

$$r(r-1)\frac{d^2\Psi_I^s}{dr^2} + \frac{d\psi}{dr} + \left[\frac{\omega^2 r^3}{r-1} - \ell(\ell+1) + \frac{s^2-1}{r}\right]\psi = 0 \tag{4.6}$$

Now, substituting $\psi = r^{1+s}(r-1)^{i\omega}y$, we get

$$\begin{aligned} &r(r-1)\tfrac{d^2y}{dr^2} + [(2s+1+i\omega)r-(2s+1)]\tfrac{dy}{dr} + \big[\omega^2 r(r-1)+ \\ &+2\omega^2(r-1)+2\omega^2-\ell(\ell+1)+s(s+1)+(2s+1)i\omega\big]\, y = 0 \end{aligned} \tag{4.7}$$

The above equation is the generalized spheroidal wave equation. Now, substituting

$$y(r) = e^{-i\sigma r} r^{-(s+1+2i\sigma)} f(u) \tag{4.8}$$

where $u = \frac{r-1}{r}$. We get by substituting,

$$u(1-u)^2\frac{d^2f}{dr^2} + \left(c_1 + c_2u + c_3u^2\right)\frac{df}{dr} + (c_4 + c_5u)\,f = 0 \tag{4.9}$$

where coefficients in terms of B_1, B_2, B_3 are

$$c_1 = B_2 + \frac{D_1}{x_0} \tag{4.10}$$
$$c_2 = -2\left[c_1 + 1 + i\left(\omega + \omega x_0\right)\right] \tag{4.11}$$
$$c_3 = c_1 + 2(1 + i\omega) \tag{4.12}$$
$$c_5 = \left(\frac{B_2}{2} + i\omega\right)\left(\frac{B_2}{2} + i\omega + 1 + \frac{B_1}{x_0}\right) \tag{4.13}$$
$$c_4 = -c_5 - \frac{B_2}{2}\left(\frac{B_2}{2} - 1\right) + \omega(i\omega) - i\omega x_0 c_1 + B_3 \tag{4.14}$$
$$= -\left(\frac{B_2}{2} + i\omega\right)\left(B_2 + \frac{B_1}{x_0}\right) - i\omega x_0 c_1 + B_3 \tag{4.15}$$

and B_1, B_2, B_3 are defined as,

$$\begin{aligned} B_1 &= -(2s+1) \\ B_2 &= 2(s+1+i\omega) \\ B_3 &= 2\omega^2 - \ell(\ell+1) + s(s+1) + (2s+1)i\omega \end{aligned}$$

Substituting a series expansion of the form:

$$f(u) = \sum_{n=0}^{\infty} a_n u^n \tag{4.16}$$

we get a three-term recursion relation for the coefficient a_j:

$$\alpha_0 a_1 + \beta_0 a_0 = 0 \tag{4.17}$$
$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n = 1, 2, \ldots \tag{4.18}$$

where α_j,β_j and γ_j are simple functions of the frequency ω, ℓ and s:

$$\begin{aligned} \alpha_n &= n^2 + (2 - 2i\omega)m + 1 - 2i\omega \\ \beta_n &= -\left[2n^2 + (2 - 8i\omega)n - 8\omega^2 - 4i\omega + \ell(\ell+1) + 1 - s^2\right] \\ \gamma_n &= n^2 - 4i\omega n - 4\omega^2 - s^2 \end{aligned} \tag{4.19}$$

The series (4.16) should be convergent and the condition which guarantees this to be true comes from the mathematical theorem due to Pincherle:

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \cdots \tag{4.20}$$

The above equation can also be shown to be equivalent to the form,

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} -} \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} -} \cdots \frac{\alpha_0 \gamma_1}{\beta_0} = \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1} -} \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} -} \cdots \tag{4.21}$$

Now, our problem boils down to solving equation(4.20) or (4.22) for ω. We can do this numerically, and get the values of ω which comes out to be complex number, as we earlier suggested. This method gives the most accurate numerical solution for the scattering problem.

4.3.1 Application: Schwarzschild metric

Leaver's method gives exact values of ω. We have computed it at different l,n values as below:
For scalar field, s=0

l	n	σ_{Leaver}
0	0	$0.38355+0.154866i$
	1	$0.247176+0.206776i$
	2	$0.231103+0.198446i$
	3	$0.221964+0.198356i$
1	0	$0.686916+0.131385i$
	1	$0.597396+0.207954i$
	2	$0.590163+0.197671i$
	3	$0.587957+0.195249i$
2	0	$0.766744+0.32998i$
	1	$0.975864+0.209931i$
	2	$0.968364+0.197383i$
	3	$0.967956+0.194602i$

For Electromagnetic field, s=1

l	n	σ_{Leaver}
0	0	$0.242061-0.0625i$
	1	$-1.37812*10^-21-1.06261*10^-21i$
	2	$-1.29273*10^-17-3.37112*10^-17i$
	3	$1.66362*10^-18-3.54525*10^-19i$
1	0	$0.579943+0.0820929i$
	1	$0.516248+0.214982i$
	2	$0.500726+0.192284i$
	3	$0.499325+0.186996i$
2	0	$0.97934+0.0877117i$
	1	$0.929004+0.21101i$
	2	$0.915742+0.19539i$
	3	$0.915651+0.191652i$

For Gravitational field, s=2

l	n	σ_{Leaver}
0	0	$0.38355+0.154866i$
	1	$0.247176+0.206776i$
	2	$0.231103+0.198446i$
	3	$0.221964+0.198356i$
1	0	$0.686916+0.131385i$
	1	$0.597396+0.207954i$
	2	$0.590163+0.197671i$
	3	$0.587957+0.195249i$
2	0	$0.766744+0.32998i$
	1	$0.975864+0.209931i$
	2	$0.968364+0.19738i$
	3	$0.967956+0.194602i$

4.4 WKB Method

Apart from Leaver's method, there exist another method to solve for approximate values of eigenvalues of radial equation (3.5), known as WKB method. Consider the equation,

$$\frac{d^2\Psi}{dx^2}+Q(x)\Psi=0 \tag{4.22}$$

which is equivalent to the master equation, where we have taken x as a tortoise coordinate. The domain of $Q(x)$ can be split in three regions: a region I to the left of the turning point where $Q(x_1)=0$, a matching region II with $x_I<x<x_{II}$, and a region III to the right of the turning point where $Q(x_{II})=0$. By standard perturbation techniques, we will solve for ψ in region I and III.

Relation with Quantum Mechanics: For a wave incident on the barrier from $x=\infty$ with a given amplitude, it is a standard calculation in quantum mechanics to determine the amplitude of the wave reflected back to $x=\infty$ and that transmitted to $x=-\infty$. If the energy is below the peak of the potential, the reflected amplitude is comparable to the incident amplitude, while the transmitted amplitude is much smaller i.e. transmitted amplitude is $e^(-\gamma)$ times the incident wave amplitude and reflected amplitude is $\sqrt{1-e^{-2\gamma}}$ times the incident wave amplitude. But for the case of black holes, there is no incident wave coming from infinity due to our boundary conditions, and it is purely outgoing at spatial infinity. Therefore, due to conservation of energy (3.19), the transmitted and reflected waves have comparable amplitudes. The only way we can have comparable amplitudes for transmitted and reflected waves is when the turning points are close to each other or when there is only one turning point. In quantum mechanics this occurs when energy is equal to the maximum value of the potential. Here, in this case we demand Q(x) maximum value to be 0 i.e. $Q(x)|_{max}=0$. However, if $Q(x)|_{max}\geq 0$ or the two turning points are close to each other, we can apply standard WKB method matching solutions of the other two regions.

Validity for WKB method to work: The WKB method gives good results when the turning points are very near to each other[7], and when the potential for the black hole system has one

peak in region II.
From the above potential we find that WKB approximation gives good approximate answers in

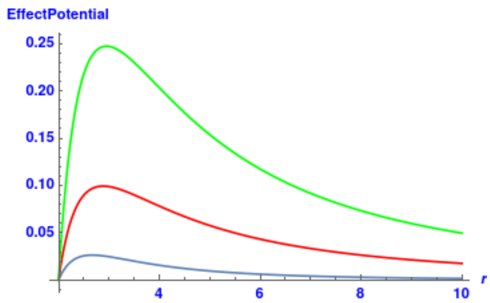

Figure 4.1: Blue ($\ell = 0$), Red ($\ell = 1$), Green ($\ell = 2$)

lower ℓ limit.
Firstly, let us find the approximate solutions in region I and III. Assuming potential varies slowly in these regions , we Introduce a small valued parameter ε in such a way that the unperturbed equation is solvable, equation 4.22 is written as

$$\varepsilon^2 \frac{d^2\psi^2}{dx^2} + Q(x)\psi = 0 \tag{4.23}$$

Taking ansatz as

$$\psi = e^{\frac{1}{\varepsilon}\sum \varepsilon^n S(x)_n} \tag{4.24}$$

Putting it in equation 4.23, we get

$$\begin{aligned} &\left(S_0' + \varepsilon S_1' + \varepsilon^2 S_2' + \dots\right)^2 + \left(\varepsilon S_0'' + \varepsilon^2 S_1'' + \varepsilon^3 S_2'' \cdots\right) + Q(x) = 0 \\ \implies &\left(S_0'^2 + \varepsilon^2 S_1'^2 + \varepsilon^4 S_2'^2 + 2\varepsilon S_0' S_1' + 2\varepsilon^2 S_0' S_2' + 2\varepsilon^3 S_0' S_3' + 2\varepsilon^3 S_1' S_2' + \dots\right) \\ &+ \left(\varepsilon S_0'' + \varepsilon^2 S_1'' + \varepsilon^3 S_2'' + \dots\right) + Q(x) = 0 \end{aligned}$$

Equating the coefficients of ε and higher powers , we get:

$$\begin{aligned} S_0'^2 + Q(x) &= 0 \\ 2S_0' s_1' + S_0'' &= 0 \\ S_1'^2 + 2S_0' S_2' + S_1'' &= 0 \end{aligned}$$

Therefore, we have shown that in region I and III, when the two turning points are close togther, the solution ψ to the first order approximation is

$$\Psi_{\mathrm{I}} \sim Q^{-1/4} \exp\left\{\pm \mathrm{i} \int_{x_2}^{x} [Q(t)]^{1/2} dt\right\} \tag{4.25}$$

$$\Psi_{\mathrm{III}} \sim Q^{-1/4} \exp\left\{\pm \mathrm{i} \int_{x}^{x_1} [Q(t)]^{1/2} dt\right\} \tag{4.26}$$

Taylor expanding $Q(x)$ around the central maximum x_0 to the second order,

$$Q(x) = Q(x_0) + \frac{1}{2} Q''\Big|_{x_0} (x - x_0)^2 + O\left((x - x_0)^3\right) \tag{4.27}$$

By change of variables

$$\begin{aligned} k &= \frac{1}{2} Q_o'' \\ t &= (4k)^{1/4} e^{i\pi/4} (x - x_0) \\ \nu + \frac{1}{2} &= -\frac{iQ_0}{\sqrt{2Q_0''}} \end{aligned}$$

(4.22) equation can be written as,

$$\frac{d^2\psi}{dt^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}t^2\right)\psi = 0 \tag{4.28}$$

whose solutions are parabolic cylinder functions, denoted by $D_\nu(z)$. Therefore, solution in region II is the superposition of these functions:

$$\Psi = AD_\nu(z) + BD_{-\nu-1}(iz), \quad z \equiv \left(2Q_0''\right)^{\frac{1}{4}} e^{i\frac{\pi}{4}} (r_* - \bar{r}_*) \tag{4.29}$$

By imposing the boundary conditions, and taking asymptotic behavior of parabolic functions we get, $1/\Gamma(-\nu) = 0$, or $\nu = 0, 1, 2, 3, \ldots$. Therefore, we get "Bohr-Sommerfield quantization rule" in the first order approximation as,

$$Q_0/\sqrt{2Q_0''} = \mathrm{i}(n + 1/2), \quad n = 0, 1, 2, \ldots \tag{4.30}$$

This condition equivalent to what we found in our standing waves discussion, and this is the condition for our modes to be discrete and complex.
From $\omega^2 = Q + V$, we get the final form of ω in terms of quantities calculated at the maxima of the potential,

$$\omega_n^2 = i\left(n + \frac{1}{2}\right)\sqrt{2Q_0''} + \left(1 - \frac{2GM}{r_0}\right)\left(\frac{\Lambda}{r_0^2} + \frac{2\beta M}{r_0^3}\right) \tag{4.31}$$

In eikonal limit, we can relate QNM to lyapunov exponent

$$\omega_{QNM} = \Omega_c l + i\left(n + \frac{1}{2}\right)|\lambda| \tag{4.32}$$

which shows that the rate of divergence of circular null geodesics at the light ring, as measured by the principal Lyapunov exponent, is equal (in the geometrical optics limit) to the damping time of black-hole perturbations induced by any massless bosonic field.

We can also go to higher orders of $Q(x)$ around its maxima in equation (4.27), and get the quantization rule in the similar way. Working out in this way upto 6th order approximation, we get the final form of *omega* in terms of quantities calculated at maxima of potential,

$$\omega^2 = \left[V_0 + \left(-2V_0''\right)^{1/2}\tilde{\Lambda}(n)\right] - \mathrm{i}\alpha\left(-2V_0''\right)^{1/2}[1 + \tilde{\Omega}(n)] \tag{4.33}$$

where,

$$\tilde{\Lambda}(n) = \frac{1}{\left(-2V_0''\right)^{1/2}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4} + \alpha^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2\left(7 + 60\alpha^2\right)\right] \tag{4.34}$$

$$
\begin{aligned}
\tilde{\Omega}(n) = \frac{1}{\left(-2V_0''\right)} & \left[\frac{5}{6912} \left(\frac{V_0'''}{V_0''} \right)^4 \left(77+188\alpha^2\right) - \frac{1}{384} \frac{V_0''^2 V_0^{(4)}}{V_0''3} \left(51+100\alpha^2\right) \right. \\
& + \frac{1}{2304} \left(\frac{V_0^{(4)}}{V_0''} \right)^2 \left(67+68\alpha^2\right) + \frac{1}{288} \frac{V_0''' V_0^{(5)}}{V_0''^2} \left(19+28\alpha^2\right) \\
& \left. - \frac{1}{288} \frac{V_0^{(6)}}{V_0''} \left(5+4\alpha^2\right) \right].
\end{aligned} \tag{4.35}
$$

An important remark is that the above formula will be valid for any spacetime metric, as every perturbed spacetime takes the form (**??**).

4.4.1 Application: Schwarzschild metric

The graph of the effective potential has one peak, and the turning points are close to each other, hence we can apply WKB method. Values of ω for different spins at different values of spherical harmonics l, overtone number are numerically calculated below:

For scalar field, s=0

l	n	σ_{WKB}
0	0	$0.104648 - 0.115196i$
	1	$0.0891898 - 0.354959i$
	2	$0.063479 - 0.594572i$
	3	$0.0255008 - 0.83504i$
	4	$0.0247885 + 1.07711i$
	5	$0.0872284 + 1.32127i$
1	0	$0.291114 - 0.0980014i$
	1	$0.262212 - 0.307432i$
	2	$0.223543 - 0.52681i$
	3	$0.173702 - 0.748629i$
2	0	$0.483211 - 0.0968049i$
	1	$0.463192 - 0.29581i$
	2	$0.43166 - 0.503433i$
	3	$0.392578 - 0.715869i$

For Electromagnetic field, s=1

l	n	σ_{WKB}
1	0	$0.291114 - 0.0980014i$
	1	$0.262212 - 0.307432i$
	2	$0.223543 - 0.526817i$
	3	$0.173702 - 0.748629i$
2	0	$0.483211 - 0.0968049i$
	1	$0.463192 - 0.29581i$
	2	$0.43166 - 0.503433i$
	3	$0.392578 - 0.715869i$
3	0	$0.675206 - 0.096512i$
	1	$0.660414 - 0.2923441i$
	2	$0.634839 - 0.494118i$
	3	$0.602182 - 0.701053i$

For gravitational field, s=2

l	n	σ_{WKB}
2	0	$0.373162 - 0.0892174i$
	1	$0.346017 - 0.274915i$
	2	$0.302935 - 0.471064i$
	3	$0.247462 - 0.672898i$
3	0	$0.599265 - 0.0927284i$
	1	$0.582355 - 0.281406i$
	2	$0.5532 - 0.476684i$
	3	$0.515747 - 0.677429i$
4	0	$0.809098 - 0.0941711i$
	1	$0.796499 - 0.284366i$
	2	$0.773636 - 0.478974i$
	3	$0.743312 - 0.6783i$

4.4.2 Application: Reissner–Nordström metric

Similar to the case of Schwarzschild metric, we numerically computed the ω for scalar field assuming $q = 0.5$

l	n	Q	σ_{WKB}
0	0	0.5	$0.247376 - 0.249018i$
	1	0.5	$0.887109 - 0.922351i$
	2	0.5	$1.37073 - 1.41889i$
	3	0.5	$0.142375 - 0.863119i$

l	n	Q	σ_{WKB}
1	0	0.5	$0.26604 - 0.145384i$
	1	0.5	$0.700671 - 0.66721i$
	2	0.5	$0.992813 - 0.97177i$
	3	0.5	$0.0604634 - 0.758635i$

l	n	Q	σ_{WKB}
0	0	0.5	$0.327777 - 0.125534i$
	1	0.5	$0.660095 - 0.584159i$
	2	0.5	$0.898499 - 0.844405i$
	3	0.5	$0.150439 - 0.733876i$

4.5 Bibliography Notes

Section 4.2 was influenced from [2], section 4.3 was influenced by [8], and section 4.4 was heavily influenced by [9].

Chapter 5

Conclusion

The late-stage ringdown phase of the gravitational waveform is often associated with the null particle orbit ("photon ring") of the black hole spacetime. We have explicitly shown that this relationship is valid only upto second order of taylor expansion of the potential function around the maximum. When the potential function is expanded to higher order, the relationship between null particle orbit and the gravitational waveform vanishes, hence showing us that the association was indeed superficial.

We have shown how unlike most idealized macroscopic physical systems, perturbed BH spacetimes are intrinsically dissipative due to the presence of an event horizon. Due to the boundary conditions QNMs have complex frequencies, the imaginary part being associated with the decay timescale of the perturbation. The corresponding eigenfunctions are usually not normalizable, and, in general, they do not form a complete set.

We have also numerically calculated the oscillation frequency of QNMs in case of Schwarszchild BH by WKB and leaver method, and of RN black hole by WKB method. By these numerical calculations we have shown how the stability of Schwarszchild and RN black holes.

In future my aim is to study vector and tensor decomposition, QNMs in case of Kerr black hole, analysis of gravitational waveform in ringdown phase and QNMs in case of compact stars.

Appendix A

Appendix A.1

A.1 Derivation of Conservation of Energy through lyapunov exponent

Theorem: For any matrix L_{ij} such that

$$\frac{dL_{ij}}{dt} = K_{im}L_j^m \tag{A.1}$$

where

$$K_{im} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

then every element of L matrix is equal.

Proof: Solving the differential equation we get the system of differential equation as

$$\begin{aligned} \frac{da}{dt} &= \frac{h}{\dot{t}}|_r c \\ \frac{db}{dt} &= \frac{h}{\dot{t}}|_r d \\ \frac{dc}{dt} &= \dot{t}^{-1}(\frac{d\frac{\partial L}{\partial r}}{dr}\Bigg)|_r a \\ \frac{dd}{dt} &= \dot{t}^{-1}(\frac{d\frac{\partial L}{\partial r}}{dr}\Bigg)|_r b \end{aligned} \tag{A.2}$$

Solving the above set of equations we find two symmetric equations showing that both will have same solutions, thus indicating that all the elements of the matrix L should be same.

In the case of particle dynamics $X = h/\dot{t}$ and $Y = \dot{t}^{-1}\frac{d\frac{\partial L}{\partial r}}{dr}$, hence by the above theorem all quantities of L_{ij} matrix are same. Assuming,

$$L_{ij} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

and solving set of equations (A.1), we get the final differential equation which we have to solve for a as:

$$\frac{d^2a}{dt^2} - \left(\frac{h}{\dot{t}^2}\frac{\partial}{\partial r}\frac{\partial L}{\partial r}\right)|_r a = 0 \tag{A.3}$$

as the coefficients are constant, thus the solution of the above equation can be written as $a = \exp(\sqrt{\gamma}t)$ where $\gamma = \left(\frac{h}{\dot{t}^2}\frac{\partial}{\partial r}\frac{\partial L}{\partial r}\right)|_r = K_1K_2$ using equation (2.18). Thus using definition of lyapunov exponent (2.14), we find

$$\lambda_0 = \sqrt{K_1 K_2} \tag{A.4}$$

which shows energy of conservation is followed.

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