classmate
Date 18 July
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L-1

(1) Prerequisite : Schutz Book
& ~~Car~~ Hartles Book

Caroll : Advance.

(2) Living Reviews in Relativity : Reviews.

(3) 4D spacetime : $(\underset{\text{manifold}}{m}, \underset{\text{metric}}{g})$

We can have manifold without metric on it
But in this course we will have manifold with metric on it.

(4) Coordinate system x^i $i = (0,1,2,3)$

Coordinate system may cover the entire manifold but typically it covers the small path & then you cover the entire manifold by covering small patches one by one.

(5) Vectors A^i : Differential operators on manifold.

In Curved manifold we can't think of a vector as arrow Bcz base pt would lie on manifold but target point is not lying on the manifold.

(6) Instead we should think of vector as tangent to the curve on the manifold lying in the tangent space at one pt.

(7) we can first have curve & then tangent vectors to it later Eg. 4-velocity.
or
we can first have vectors & then curves would be integrated as integral curve associated with that vector.

(8) ~~You can~~ in Manifold
Addition of vectors at different points is not a vectorial qty.
We have to add vector which are at same pt. / lives in same Tangent space.

See ch-1 Eric — The operation of integeration of vector field on Manifold ∄

(9) We can only ~~..~~ Integerate Scalars. We cannot Integerate vector fields & Tensor fields.

(9), (10), (11)

~~As~~ they are not a vectorial / Tensorial qty.
Bcz.

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^{i}} v^{i}$$ → Different at different pt.

(10) Dual vector (1 form) V_i

The operations which act on v^i & give $\mathbb{R}$

They are the linear operations on vectors which yield $\mathbb{R}$.

$V_i V^i = g\{\vec{V}, \vec{V}\}$

(11) If we don't have metric then there is no Relationship B/w vector & Dual vector.
2 form → 2 index object with antisym.
3 form → 3 index fully antisym.

(12) Tensors

Linear ~~Maps~~ operation on ~~B/w~~ vectors or one forms into $\mathbb{R}$. eg. $T^{\alpha\beta} p_\alpha p_\beta = \#$

(13) It is important to know if the ~~tensor~~ qty is Tensor or not.
~~But~~ one have to check if the components transform in a particular way/not.

But
there are other ways of testing if qty is tensor or not.

how is the change in coord.? ← What does it really mean when $T^{\alpha'\beta'} = \partial_\alpha x^{\alpha'} \partial_\beta x^{\beta'} T^{\alpha\beta}$

(14) Metric Tensor

Metric Tensor on manifold makes it a metric space & then we are able to calculate Distances in spacetime.

Def: $ds^2 = g_{ij}\, dx^i dx^j$ = spacetime Interval

Note of g_{ij}: (1) Convert coordinate increment to physical distance. Coordinate increment depends on coord. system. But dl^2 is scalar ind. of coord system information.

(2) Contains gravitational information.

We get Both information from g_{ij}.

$\boxed{\text{No other theory (EM) has this feature.}}$ → $\boxed{\text{All due to Equivalence Principle.}}$

(15) In Metric Space

$$A^\alpha = g^{\alpha\beta} p_\beta$$

$$p_\beta = g_{\beta\gamma} A^\gamma$$

vectors & one forms become equivalent in metric space.

But they are distinct in space which is not metric space.

(16) Inverse Metric $g^{\alpha\beta}$

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$$

~~(17) Connection will convert of metric space to affine space.~~

(18) We can have manifold without metric & connection

We can have manifold with metric & without connection.

We can have manifold without metric & with connection.

(18) If we have only connection then manifold is Affine manifold

1) Demande $\Gamma^a_{bc} = \Gamma^a_{cb}$ $\quad \forall\, b, c, a$

$\nabla_c g_{ab} = 0$ $\quad \forall\, a, b, c$

2) $\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^i_{ca} g_{ib} - \Gamma^i_{cb} g_{ai} = 0$

$\partial_c g_{ab} = \Gamma^i_{ca} g_{ib} + \Gamma^i_{cb} g_{ai}$

$\nabla_b g_{ac} = \partial_b g_{ac} - \Gamma^i_{ba} g_{ic} - \Gamma^i_{bc} g_{ai}$

$\partial_b g_{ac} = \Gamma^i_{ba} g_{ic} + \Gamma^i_{bc} g_{ai}$

$\partial_a g_{bc} = \Gamma^i_{ba} g_{ic} + \Gamma^i_{ac} g_{bi}$

$-\partial_c g_{ab} + \partial_b g_{ac} + \partial_a g_{bc} = 2\,\Gamma^i_{ab}\, g_{ic}$

$$\boxed{\Gamma^l_{ab} = \frac{g^{cl}}{2} \left(-\partial_c g_{ab} + \partial_b g_{ac} + \partial_a g_{bc}\right)}$$

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(3) $\cancel{\Gamma^a_{\ b} = \lambda^a \partial_b \phi - \delta^a_b k}$

$$\begin{tikzcd}
m \arrow[r] \arrow[d] & \Gamma \text{ (Affine manifold)} \arrow[d] \\
g \text{ (metric Space)} \arrow[r] & \begin{array}{c} \Gamma_{ab} = \Gamma_{ba} \\ \nabla_a g_{ij} = 0 \end{array}
\end{tikzcd}$$

If we have metric on manifold then it is metric space.

Metric & connection can be two different structures. when dealing with alternate gravity formulation, we try to exploit this.

Demanding $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$ ~~It is Identity~~ ⟹ ~~we don't have to demand~~

metric compatible : $\nabla_c g_{\alpha\beta} = 0$

$$\Rightarrow \Gamma^{\alpha}_{\beta\gamma} = \frac{g^{\alpha\mu}}{2}\left(-\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta}\right)$$

We could have choosen different demands & ∴ Different connection.

Ex: In theory involving torsion

$$\Gamma^{\alpha}_{\beta\gamma} = -\Gamma^{\alpha}_{\gamma\beta}$$

Connections gives us the way to differentiate

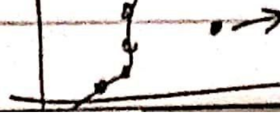
i.e. It gives us the way to // transport vector so that we can compare two vectors at one pt.

∴

Covariant differentiation requires connection to be defined on the manifold unlike Lie Diff.

① Vector fields only on Curves } Distinguish
② vector fields all over space. } B/w these Cases

① case Vector is defined all along curve



→ Not defined

In (1) case, I can differentiate along the curve only.

(25) Covariant derivative along the curve:

$$\frac{DA^\alpha}{d\lambda} = \frac{dA^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\gamma}{d\lambda} A^\beta$$

(26) Covariant Derivative along all possible direction

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\beta\gamma} A^\gamma$$

along any direction

$$\frac{DA^\alpha}{d\lambda} = u^\gamma \partial_\gamma A^\alpha + \Gamma^\alpha_{\beta\gamma} A^\beta u^\gamma$$

$$= \underbrace{(\nabla_\gamma A^\alpha) u^\gamma}$$

$\implies$ Directional Derivative of vector along the curve

$$\underbrace{\nabla_\gamma A^\alpha}$$

$\implies$ Directional Derivative of vector along any direction.

(27) If only vector along the curve is known; then $\nabla_\gamma A^\alpha$ is not defined

L-2

① **Definition** : Geodesic

Curve which extremizes distance B/w 2 points.

freely falling observer ≡ Timelike geodesic

② **Definition** : Timelike Geodesic

Timelike curve in Spacetime which extremizes proper time B/w 2 events.

③ $d\tau^2 = -ds^2 = g_{ij}\, dx^i dx^j$

$$\tau(A \to B) = \int_A^B \sqrt{g_{ij}\, dx^i dx^j}$$

$$A = \int_{q_1 t_1}^{q_2 t_2} L(q,\dot{q},t)\, dt$$

$$A[q], q_1 t_1, q_2 t_2$$

Proper time is functional of the path just like $A[q]$

$$A = m \int d\tau = m \int \sqrt{g_{ij}\, dx^i dx^j}$$

Action is functional of path $A[q]$

④ Because

g_{ij} depends on path we are using.

dx^i depends on the path we are using

④ $$A[q(t); q_1, t_1; q_2 t_2] = \int_{q_1 t_1}^{q_2 t_2} L(q,\dot{q},t)\, dt$$

Action as a function of q_1, t_1 & q_2, t_2
(When classical Action is considered)

$$A_c(q_1 t_1 ; q_2 t_2) = \left.\frac{\partial L}{\partial \dot{q}} \delta q \right|_{q_1 t_1}^{q_2 t_2}$$

Now fixing the first end pt. q_1
we get $\delta A_c(q_2, t_2) = \frac{\partial L}{\partial \dot{q}} \delta q_2$

(5) Action is the functional of $q(t)$ with end points fixed.

$$A[q(t)] = \int dt\, \delta q \underbrace{\left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right\}}_{\text{Depends on path}}$$

δq → Depends on path.

(6) Action as a function of $(x_1, t_1; x_2, t_2)$ in

$$\delta A_c = m \int_{x_1 t_1}^{x_2 t_2} d\tau \left(\frac{d u_i}{d\tau} \right) \delta x^i$$

~~Now fixing the first end pt. fixed~~

$$\delta A = -m \int_{x_1 t_1}^{x_2 t_2} d(u_i \delta x^i)$$

Now fixing the first end pt. fixed

$$\delta A = -m (u_i \delta x^i) \Big|_{x_1 t_1}^{x_2 t_2} = -m\, u_i\, \delta x_2^i$$

$$\frac{\delta A}{\delta x^i} = -m u_i$$

~~(7) Action as a functional of $[x^i]$~~

$$\cancel{\delta A[x^i] = m\int \frac{du_i}{d\tau}\,\delta x^i\, d\tau}$$

$$\cancel{\frac{du_i}{d\tau} = 0}$$

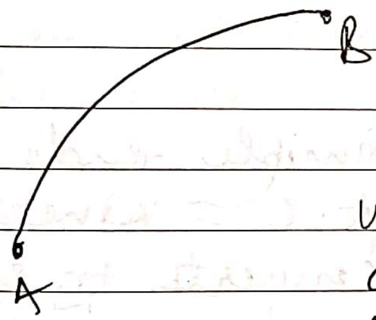
~~Depends on path.~~

~~Depends on path~~

Doubt

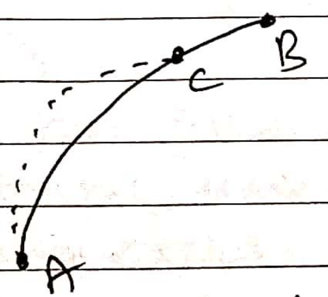
(5) Extremizes means maximizing proper time. Why?

(6) It is max. for short seperation.
But For larger seperation max. turns to a saddle point.



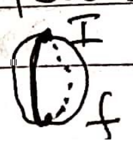
What happens if there is another geodesic & cuts the original geodesic?

There is apossibility that a nearby geodesic doesn't cut the original one.
Thus, it is of no use.



But if it cuts.

Example
Sphere: Geodesics = Great Circle



C: Conjugate pt.

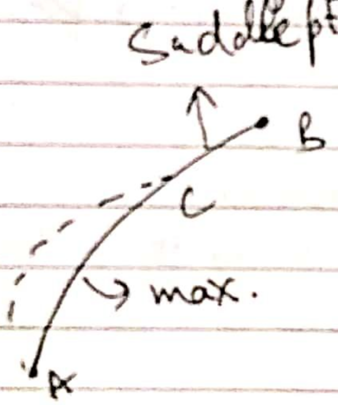
conjugate point; Singularity theorem

(7) Prove: Proper time from A → C in Original geodesic = Proper time from A → C in nearby Geodesic to atleast second order.

(8) Prove: Proper time from A → C in Original geodesic will have max. proper time.
Beyond Conjugate pt. we can have worldline which will have larger proper time than Original geodesic.

Ex. go along nearby geodesic & follow C → B

Doubt



Contrary in CM:

(9) Variation of Action principle tends to be minimum upto certain pt. C = kinectic focus & further minimum converts to saddle pt.

Doubt

(10) $$\tau = \int_A^B \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda$$

$$= \int_A^B L \, d\lambda$$

Putting in EOM

$$\frac{\partial L}{\partial x^i} = \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^i}$$

$$\Rightarrow \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = K \frac{dx^i}{d\lambda}$$

$$K = \frac{d \ln L}{d\lambda}$$

RHS is Tangent to Geodesic.

(11)

(12) Action is reparameterization invariant. Therefore we can change to any parametrization of the curve.

(13) when $\lambda = \tau \Rightarrow d\lambda = d\tau$

$$L = \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} = 1$$

$\therefore K = 0$

(14) $K = 0$ holds for all transformations

$$\lambda = a\tau + b$$

All the parameter on the geodesic related to proper time linearly are Affine Parameters.

(15) For Affine Parameter

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

(16) Example of non affine parameters:
Sometimes it is useful to take non affine par

Ex: ~~For~~ Expanding Universe

Freedman Robbinson walker metric

$$ds^2 = +dt^2 - a^2(t)\underbrace{(dx^2 + dy^2 + dz^2)}_{\text{flat Universe}}$$

a ↑ Scale factor depending on time

$$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{\dot{a}}{a}$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}$$

$$\cdot \equiv \frac{d}{dt}$$

(17) t : time interval which cosmological Observer would use

Cosmological Observer : All those Observer that are addressed relative to the cosmological fluid which is expanding.

(18) Time like Geodesic : Cosmological Observer
if x, y, z = const.
then $ds^2 = dt^2$

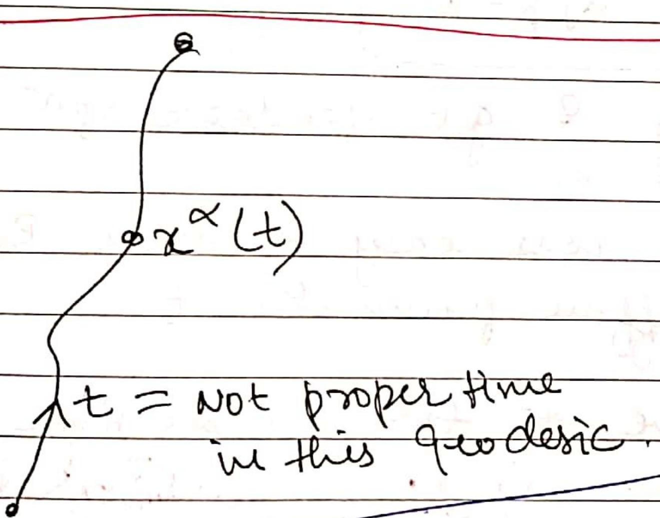
$\Rightarrow$ Now let another observer moving w.r.t. cosmological observer in x direction.

$\Rightarrow$ t is the proper time for cosmological observer, which are not moving in spatial coordinate $dx, dy = 0$

t is not the proper time for observer we are talking here.

$\Rightarrow$ Now here for the geodesic pick t as parameter because I know in this parameter I know the scale factor $a(t)$.

(19) $\therefore$ $\boxed{x^\alpha(t) = [t, x(t), 0, 0]}$ $\Rightarrow$ How? ①



for Cosmo. Obser
$x^\alpha(t) = [t, 0, 0, 0]$
$L = \int g_{ij}\, \dot{x}^i \dot{x}^j$
$= \sqrt{g_{00}} = 1$
$\therefore$ Proper time

(20) $L = \int g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} = \sqrt{1 - a^2 \dot{x}^2} \neq 1$

$\therefore$ t is not the proper time

(21) Putting in E-L eq. $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$

$\therefore \frac{\partial L}{\partial \dot{x}^i} = \text{const}$

∴ Momentum in x-direction is constant.
This is due to the fact that metric doesn't depend on the coordinates. It is only dependent on time.

$$\frac{\partial L}{\partial \dot{x}} \equiv p = \frac{1}{2L}(-2a^2\dot{x})$$

To get rid of -ve sign

$$\frac{\partial L}{\partial \dot{x}} = -p \Rightarrow p = \frac{a^2\dot{x}}{L} = \frac{a^2\dot{x}}{\sqrt{1-a^2\dot{x}^2}}$$

$$\Rightarrow 1 - a^2\dot{x}^2 = \frac{a^4\dot{x}^2}{p^2}$$

$$\boxed{\dot{x} = \frac{p}{a\sqrt{p^2+a^2}}} \quad \text{(1)}$$

$$\Rightarrow \frac{\pm p}{a\sqrt{p^2+a^2}} = \dot{x}$$

& a is function of t

Solve for x & get Geodesic eqn.

(22) This problem was easy to solve bcz we used non affine parameter t.

(23) Now we have to take proper time τ as the parameter.

$$\int L\,dt = \int \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt = \int \sqrt{g_{ij}dx^i dx^j} = \int \sqrt{g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}\,d\tau$$

from $\frac{\partial L}{\partial \dot{x}}$ / ∂L

$$d\tau = L\,dt$$

$$\frac{d\tau}{dt} = \sqrt{1-a^2\dot{x}^2} = \frac{a}{\sqrt{p^2+a^2}} \quad \text{(2)}$$

putting $\dot{x}$

solve for τ.

$$\therefore L\,dt = d\tau$$

$$= \int d\tau$$

Now if we use τ as parameter

$$\frac{dt}{d\tau} = \frac{\sqrt{p^2+a^2}}{a} \quad \text{from (1) \& (2)}$$

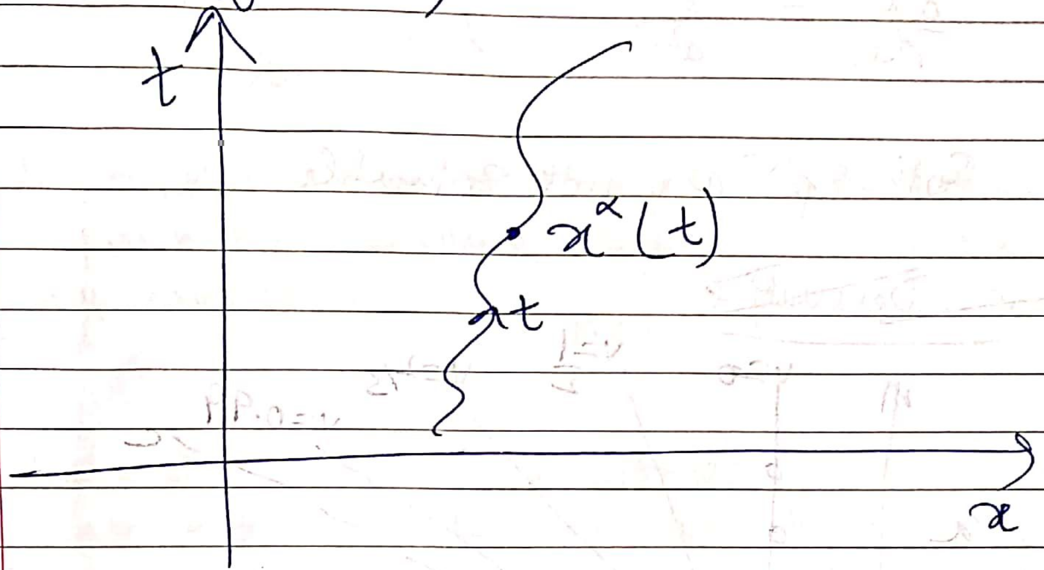
cosmological Damping! ← $\left\{\frac{dx}{dt}\right. = \frac{p}{|a^2|}$ → Damping factor we know $t = f(\tau)$

$\therefore \frac{dx}{d\tau} = \frac{p^2}{a^2(t)} = \frac{p^2}{a^2(f(\tau))}$ & then solve — How?

Now RHS is f^n of t

$\therefore$ Integral is tricky to solve — But we can solve it

(24) $\therefore$ There is a benefit of using non affine parameter t rather than using τ (proper time (affine par))



Compare with where we derived Lorentz Transf^n.

~~& as~~ (25) Here we used t because $a(t)$ was given.

(26) Now in the previous example If we have null geodesic $\therefore$ Taking t (Non affine) as the parameter Taking the limit to speed c

$L \longrightarrow 0 \longrightarrow$ as Null case $dx_i dx_i = 0$

as $L = \int \frac{dx_i}{dt}\frac{dx^i}{dt} = 0$

$p = \frac{a^2 \dot{x}}{L} \Rightarrow p \rightarrow \infty \Rightarrow \dot{x} = \frac{p}{a\sqrt{p^2+a^2}}$

$\Rightarrow \dot{x} = \frac{p}{pa\sqrt{1+(\frac{a}{p})^2}} \Rightarrow \dot{x} = \frac{1}{a} \Rightarrow x = \frac{t}{a}$

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(27) But now if we take proper time τ as the parameter.

$$\frac{dt}{d\tau} = \frac{\sqrt{p^2+a^2}}{a}$$

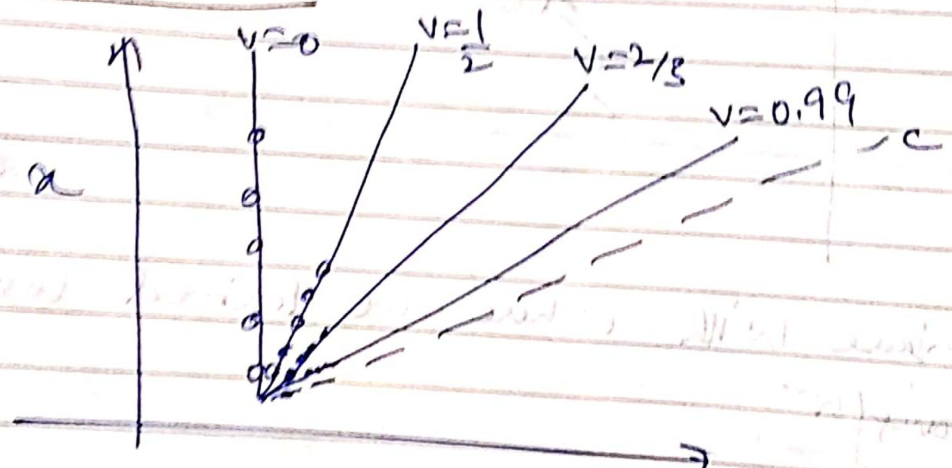
$$\frac{dx}{d\tau} = \frac{p}{a^2}$$

Taking $p \to \infty$.

Both Eq^n are not solueable

(22) ~~Lie Derivative~~

(22)



$d\tau = dt\sqrt{1-\frac{v^2}{c^2}}$

for all cases including null, if the geodesic is given in any parametrization, it is possible to describe geodesic with affine parametriza

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(7)

L-3

1. Lie derivatives $\mathcal{L}_u A^\alpha$ is Tensor
   → 2 vectors subtract at same Tangent Space
   → By explicit calculation.

2. killing vector: A vector field ξ^a s.t.

$$\mathcal{L}_{\xi} g_{ij} = 0$$

3. Examples: Translational time symmetry, Symmetry around rotation, spatial translat symmetry.
   ↳ Axial symmetry.

4. Largest no. of symmetries in spacetime = 10
   De Sitter Space has all these symmetries.
   4 Translational, 3 Rotational, 3 Boosts.

   Minkowski spacetime also has max sym. in Spacetime.
   ∴ Max. we can have 10 killing vectors.

5. Spherical Symmetry: We have 3 killing vectors to characterise Rotation.
   Static & spherical: Time translat killing vector + 3 Rot. Sym. = 4

6. If in a question symmetries are not easily seen than find solⁿ of $\nabla_i \xi_j + \nabla_j \xi_i = 0$

7. Static, Spherically symmetric spacetime (Eg. Schwarchild) $ds^2 = -f\,dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + s\theta\, d\phi^2)$

$$f = 1 - \frac{2M}{r}$$ Metric is ind. of t, ϕ

$\nabla_\alpha(\quad)$

$\therefore$ 2 obvious killing vectors

ξ_t , ξ_ϕ

This tells we can rotate our spacetime in particular direction (Axial Symmetry)

& further there are two more killing vectors which tell that we can rotate in other 2 directions as well. (full spherical symm)

→ How to know these also $\exists$?

(8) With the help of killing vectors we can get constants of motion.

Just as in Lag mechanics conservation of momentum is manifestation of spatial symmetry

here also killing vectors leads to conservation when geodesic is timelike & material particles moves along it.

If there is external force eg. EM which is stopping particle to move in geodesic then constant of motions can't be found unless those forces

(9) $\Rightarrow U^i \xi_i^{(t)} = +\tilde{E} = E/m$

Timelike

→ Timelike $\therefore$ To make Energy +ve sign is +ve

Why?

We have to divide by mass because

if $p^i \xi_i^{(t)} = E$

$\Rightarrow U^i \xi_i^{(\phi)} = +\tilde{L} = L/m$

Spacelike

To make Angular

Ang. mom. can be +ve or -ve

No assigned sign.

Bcz $U^i U_i = 1$ for ~~Time like~~ Geodesic

But to associate energy we take always timelike

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(10) Two notions of Energy.
There are many notions of Energy. But we here discuss only 2.

1. Conserved Energy $\equiv$ Killing energy $\equiv$ Energy at ∞.

$$\tilde{E} = u_\alpha \xi^\alpha_{(t)}$$

Particle at ∞ in spacetime
$\therefore$ Spacetime is flat
Because Sch. metric is asymptotically flat

$\therefore$ In flat Spacetime

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma \frac{dx^\alpha}{dt}$$

$$u^\alpha = \gamma\left(1, \frac{dr}{dt}, 0, 0\right) \qquad \xi^\alpha_{(t)} = (1, 0, 0, 0)$$

$$\tilde{E} = u_\alpha \xi^\alpha_{(t)} = \gamma$$

$\therefore$ ~~to~~ at ∞ (flat) $\tilde{E} = \gamma$
But $\tilde{E}$ is const. of motion
$\therefore$ That particle which is thrown from ∞, will have $\tilde{E} = \gamma$ throught its motion.

also obey same those symmetries.

$\boxed{\text{Only if particle is at } \infty \text{ then only it can measure } \tilde{E} = \gamma}$

2. Locally measured energy by observer in spacetime.
particle velocity u^α
Observer velocity u^α_{obs}

$\tilde{E} = u^{0'}$?

$$\boxed{\tilde{E}_{local} = u^\alpha_{obs} u_\alpha} \rightarrow \text{Verify this is the Energy By going to LIF}$$

in Obs frame
$u_{obs} = (1, 0, 0, 0)$
$\therefore \tilde{E} = u_{0'}$
$P^\alpha_{\text{part}} = (E, m\vec{v})$ justify

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$\tilde{E}_{local} \neq \tilde{E}$

$\tilde{E}$: we have to go to ∞ to measure $\tilde{E}$, ~~too~~ And Regardless if we are at ∞ or not $\tilde{E}$ will be the energy of the particle, the point is to measure it directly going upto ∞ / indirectly.

$\tilde{E}_{local}$: Just go there & measure the Energy of the particle.
the value changes with the motion in the spacetime.

(11) $\tilde{E}$ can be -ve
But $\tilde{E}_{local}$ will always be +ve

Particles in Ergosphere can have -ve killing energy. Those particles can never escape to ∞

Why?

Killing Energy can be -ve.

(2) Why $\tilde{E} \neq \tilde{E}_{local}$

Energy Redshifted?

(13) Similar Distinction happens in Angular Momentum.
But there is no confusion of sign as Ang. Mom. can take any sign.

why always −ve ?

(14) $g = |g_{ab}| < 0 \quad \therefore \sqrt{-g}$

$\boxed{\text{But why −ve ?}}$

(15) Levi-Civita Tensor

⇒ $[a\ b\ c\ d]$ is not a Tensor.
as $[a\ b\ c\ d] = [a'\ b'\ c'\ d']$
⇒ $\epsilon_{abcd} = \sqrt{-g}\ [a\ b\ c\ d]$ is a Tensor.

(16) When you integerate over manifold, you integerate over the form
Imp If you Integrate over 4D manifold, you integrate over 4-form, which is full A.S. with 4 indices.

ϵ_{abcd} is the volume forme in 4D Spacetime.

$\sqrt{-g}$ comes Because ϵ_{abcd} is the vol. form.

(17) Curvature Tensor

$$\boxed{\text{for scalars : } \nabla_i \nabla_j f - \nabla_j \nabla_i f = 0}$$

for vectors & Tensors They do not commute & brings in R term.

$$\nabla_i \nabla_j A^\mu - \nabla_j \nabla_i A^\mu = -R^\mu{}_{\nu i j} A^\nu$$

$$\nabla_i \nabla_j A_\mu - \nabla_j \nabla_i A_\mu = + R^\nu{}_{\mu i j} A_\nu$$

$\nabla_i \nabla_j T^{ab} - \nabla_j \nabla_i T^{ab} =$

(18) Metric g_{ij} is the starting point.

$$d\ell^2 = g_{ij}\, dx^i\, dx^j$$

But g_{ij} is ambiguous as it carries gravity info + coordinate info.

But

R' carries true info of Gravity

as $R \neq 0 \Rightarrow$ Gravity

$R = 0 \Rightarrow$ No Gravity

(19) If our assumption of $\nabla_i g_{ab} = 0$

$\Gamma_{ij} = \Gamma_{ji}$

is not valid

than Γ & g are not connected

But if on a manifold, either connection is defined or metric is defined.

R^e_{ijk} is defined with connection.

& not on metric.

If metric is not defined still R can

(20) In 4D $R^a{}_{bcd}$ has 20 Ind. comp.
In 3D $\Rightarrow$ 6 Ind comp, In 2D = 1 Ind comp.

(21) Can calculate $R^a{}_{bcd}$ computation
in Maple download GR Tensor.

(22) The only possibility of contraction on $R^a{}_{bcd}$ is 1 & 3
as

$$\underbrace{g^{ab}}_{S} \underbrace{R_{abcd}}_{A.S} = 0$$

$$1\&3 = -1\&4 = 2\&4 = -2\&3$$

(23) $R_{je} = R^a{}_{jae}$

$$R_{je} = g^{am}(\partial_m g$$

$$R_{je} = \partial_a \Gamma^a_{je} - \partial_e \Gamma^a_{ja} + \Gamma^a_{ao}\Gamma^o_{ej} - \Gamma^a_{eo}\Gamma^o_{aj}$$

$$= \partial_a \Gamma^a_{je} + \Gamma^a_{ao}\Gamma^o_{ej} - \partial_e \partial_j \ln(\sqrt{-g}) - \Gamma^a_{eo}\Gamma^o_{aj}$$

In $(\nabla_i \nabla_j - \nabla_j \nabla_i)A^a$
$\Rightarrow$ we did not use g_{ab} just by Γ we derived R

$$R = R_{je} g^{je}$$

$$R = R_{ej} g^{ej} = R_{ej} g^{je}$$

$$R_{je} g^{je} - R_{ej} g^{je} = 0$$

$$g^{je}(R_{je} - R_{ej}) = 0 \Rightarrow R_{je} = R_{ej} \therefore \text{Sym}$$

(24) R_{jl} has ${}^5C_2$ comp = 10

from 20 we got 10

other 10 are in weyl Tensor.

$\therefore$ Reiman Tensor = Ricci + weyl.

Ricci $\equiv$ (Trace of Reimann Tensor)

weyl: (trace free part of Rieman Tensor)

(25) R has 1 Ind. as it is Scalar

(26) Ricci Tensor contains partial Info about gravity

Riem. Tensor Contains full Info.

(27) Bianchi Identies

$$\nabla_{[i} R^{ab}{}_{cd]} = 0$$

$$g^a_c \, g^i_b \left(\nabla_{[i} R^{ab}{}_{cd]} \right) \Rightarrow \nabla_\beta G^{ab} = 0$$

$$G_{ab} = R_{ab} - \frac{g_{ab} R}{2} = \text{Einstein Tensor}$$

Einstein didn't know this identity & $\therefore$ he did not know what to take LHS of

→ Einstein field Eq^n

→ Einstein Tensor $G_{ab} \Rightarrow$ Symmetric $\Rightarrow$ 10 Ind comp

(28) Einstein field Eq^n

$$G^{ab} = 8\pi k T^{ab}$$

Geometry (G^{ab}), matter (T^{ab})

Bianchi Identity is from geometry

∴ Consequence of Bianchi Identity

$\nabla_b T^{ab} = 0$ [Energy Mom Cons]

∴ first comes Geometry → field Eq^n → Energy Mom Conservation.

(29) ⟹ We can also think $\nabla^a T_{ab} = 0$ as basic. And this is the correct way of Thinking Because we Earlier show Bianchi Identity is True & Now due to field Eq^n ⟹ $\nabla_a T^{ab} = 0$

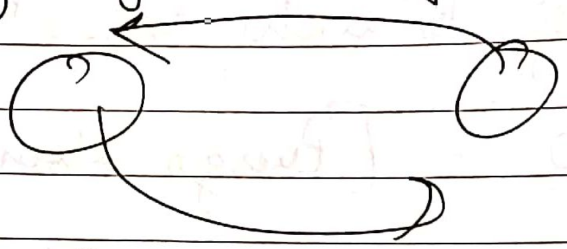
But if the field Eq^n change, then Bianchi Ident $\not\Rightarrow$ $\nabla_a T^{ab} = 0$

∴ Think $\nabla_a T^{ab} = 0$ due to general Covariance (Paddy).

(30) Schwarchild is the exact sol^n to EF Eq^n
There are tons of exact sol^n but very few relevant.
in higher dim. exact sol^n increases.

(31) When we find exact solution to EF Eqn we find for the whole spacetime We have ~~find~~ describe motion at all times from $-\infty \longrightarrow \infty$.

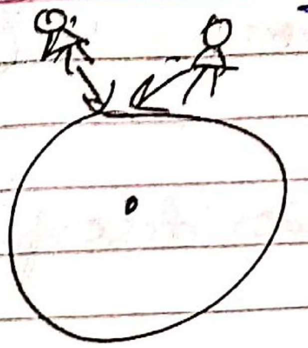
Eg. Dynamics of 2 Rotating body.



(32) ∴ EF eqn has to describe gravitation collapse of Nebulae to stars, orbital motion, feature of orbital motion is they emit grav. wave ∴ describe grav. wave, As they emit grav. wave, orbital gets shrink & then Black Body.

But we can't find all of this ∴ we shrunk spacetime to finite time as approx. and get solution. to orbital motion, Black hole etc.

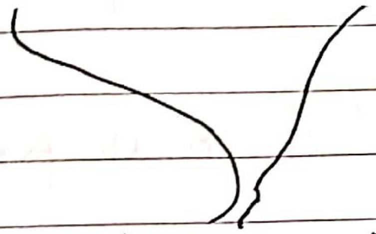
(33) But we can't find solution for all spacetime ~~from -∞ to~~ rather than for finite time.



The gravitational field is not Uniform ∴ These two come near each other

∴ $dF \neq 0$

In GR sense, Geodesic Deviation is there



Geod Devi VS Symmetric Spacetime

& ∴ $R \neq 0$ ⇒ There is no // Transp ⇒ Curvature ~~Tensor~~ is there.

if the grav. field is Uniform $dF = 0$

i.e.

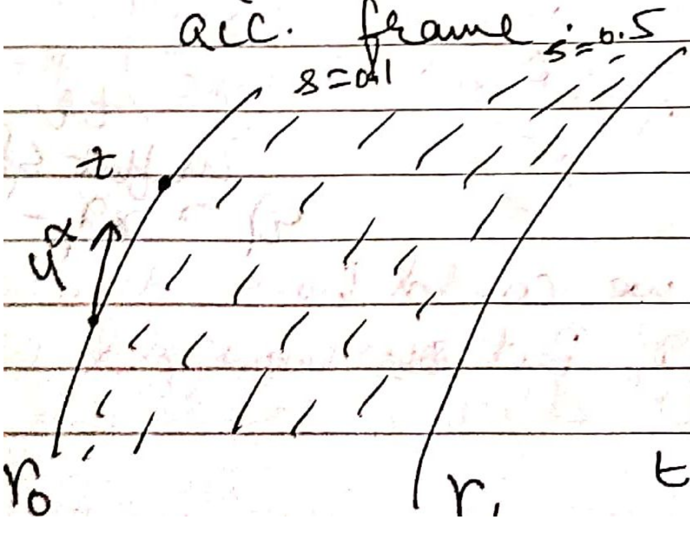
in GR way

Geodesic deviation is $0 \Rightarrow R = 0 \Rightarrow$

// Transp ⇒ Curvature $= 0$

∴ Flat spacetime.

Equivalent to Coordinate transf^n to acc. frame



sequence of geodesics $\gamma(s)$ s.t.

$\gamma(s=0) = \gamma_0$

$\gamma(s=1) = \gamma$

s : parameter that label each geodesic

t : Running parameter on each $\gamma(s)$

Assuming t is affine parameter s.t. Geodesic eqn has its usual form.

$x^\alpha = x^\alpha(s, t)$ Parametric description of $\gamma(s)$

s → selecting geodesic

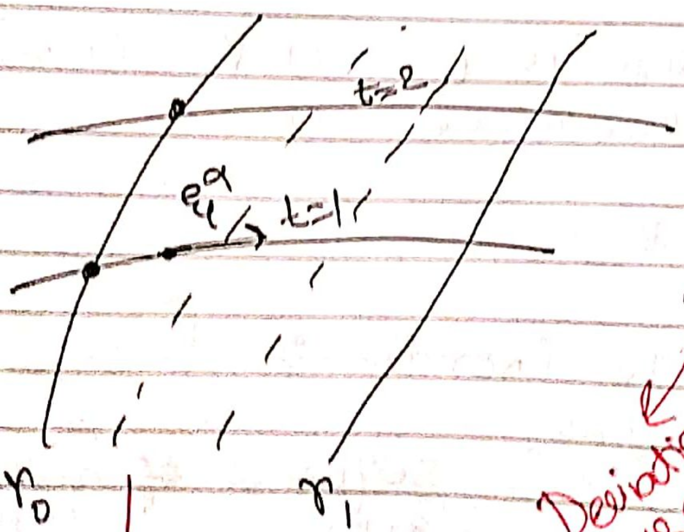
t → selecting point on selected geodesic.

(4) $u^\alpha = \left(\dfrac{\partial x^\alpha}{\partial t}\right)_s$ ∴ s is fixed

∴ As t is Affine parameter & $\gamma(s)$ is geodesic

∴ $u^\beta \nabla_\beta u^\alpha = 0$

(5) Keeping t fixed ~~move~~ varying s in $x^\alpha(s,t)$



We define family of Cross Curves that run from $\gamma_0 \to \gamma_1$

ξ^α = Tangent vector to all Cross Curves

Deviation vector $\equiv \left(\dfrac{\partial x^\alpha}{\partial s}\right)_t$ &

in flat spacetime $\xi^\alpha = x^\alpha_1 - x^\alpha_0$

In flat spacetime we could have drawn vector from γ_0 to γ_1. But on manifold we can't.

(6) $\partial_\beta U^\alpha \xi_\phi^\beta$ = Directional derivative of U^α along ξ_ϕ

$$= \left(\frac{\partial U^\alpha}{\partial s}\right)_t = \frac{\partial^2 x^\alpha}{\partial s \partial t}$$

$\partial_\beta \xi_\phi^\alpha U^\beta$ = Directional derivative of ξ_ϕ^α along U^β

$$= \left(\frac{\partial \xi_\phi^\alpha}{\partial t}\right)_s = \frac{\partial^2 x^\alpha}{\partial s \partial t}$$

$$\mathcal{L}_U \xi_\phi^\alpha = \partial_\beta \xi_\phi^\alpha U^\beta - \xi_\phi^\beta \partial_\beta U^\alpha = 0$$

$\therefore \mathcal{L}_U \xi_\phi^\alpha = -\mathcal{L}_{\xi_\phi} U^\alpha = 0$ ← using Lie Derivative

$\therefore U^\beta \nabla_\beta \xi_\phi^\alpha = \xi_\phi^\beta \nabla_\beta U^\alpha$

$\therefore$ They are Lie Transported

This Result comes just from Construction.

(7) $\frac{d}{dt}(\xi_i U^i) = \frac{D}{dt}(\xi_\alpha U^\alpha) = U^\beta \nabla_\beta \xi_\alpha U^\alpha + \xi_\alpha U^\beta \nabla_\beta U^\alpha$

$$= (\nabla_\beta U_\alpha)\, \xi_\phi^\beta U^\alpha$$

there we were using $\nabla_i g_{ij} = 0$ i.e. $\nabla_\gamma g_{\alpha\beta}$ of lie vector ξ_i ... $\nabla_i U_j$

Result of Lie Derivative compare

$$= \frac{\nabla_\beta}{2}(U_\alpha U^\alpha)\, \xi_\phi^\beta$$ constant as we are using affine parameter

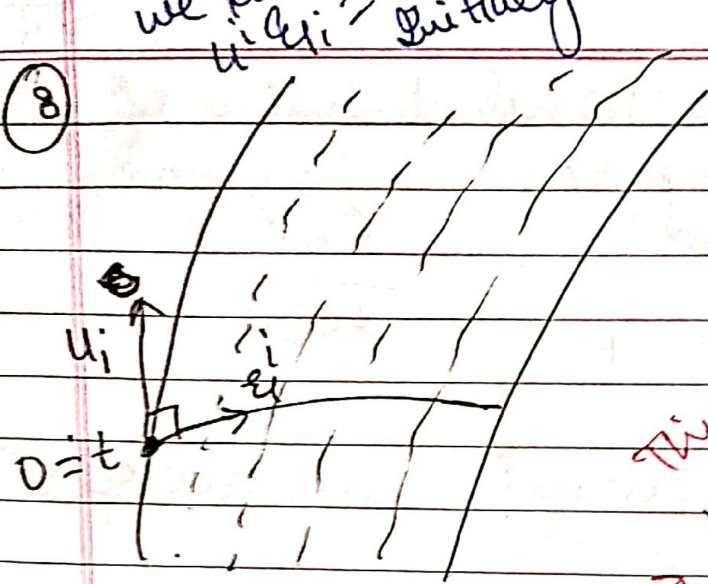
$= 0$ → Assuming ~~Time like~~ Geod. parameter Affine

Killing eqn ⇒ $\therefore \xi_i U^i = \text{const.}$

All these Results comes just from the setup

But from (7) we know that $u^i\xi_i = \text{const}$ ∴ if $u^i\xi_i = 0$ initially it will remain so.

(8)



This Argument guarantees $u^i\xi_i = 0$ at Initial $t = 0$. But what about $t = 2$ pts B.C. there curves are not arbitrary Rather fixed due to $u^\alpha = \frac{dx^\alpha}{dt}$

setting origin t parameter on each geodesic ~~s.t.~~ which can be arbitrary.

∴ Set $t = 0$ parameter on each Curve s.t. $\xi^i_i u_i = 0$

i.e. Curves are orthogonal

(9) from (7) $\frac{d}{dt}(\xi_{ii}u^i) = 0$

As I move along geodesic $\xi_{ii}u^i = \text{const}$

But Not along Cross Curves

Formal proof: of $u^i\xi_{ii} = 0$ (Indep. of making Curve I^n)

(10) ξ^α_i can be decomposed as

$$\boxed{\begin{aligned} &\xi^\alpha_i = \lambda u^\alpha + \bar{\xi}^\alpha_i \\ &\text{s.t.} \\ &\lambda = +\xi^\alpha_i u_\alpha \\ &u_\alpha \bar{\xi}^\alpha_i = 0 \end{aligned}}$$

But by (7) $\lambda = \text{const}$

∴ Now we have to prove ξ^α_i can be replaced by $\bar{\xi}^\alpha_i$

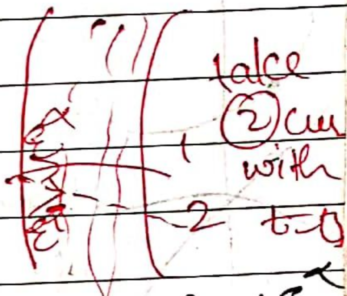
To prove this
we have to show that $\bar{\xi}^\alpha$ holds all the properties which ξ^α holds.

i.e. T.P. $\mathcal{L}_u \bar{\xi}^\alpha = 0$

Proof:
$$u^\beta \nabla_\beta \bar{\xi}^\alpha = u^\beta \nabla_\beta (\xi^\alpha - \lambda u^\alpha)$$
$$= u^\beta \nabla_\beta \xi^\alpha - \lambda u^\beta \nabla_\beta u^\alpha$$
$$= u^\beta \nabla_\beta \xi^\alpha$$

$$\bar{\xi}^\beta \nabla_\beta u^\alpha = (\xi^\beta - \lambda u^\beta) \nabla_\beta u^\alpha$$
$$= \xi^\beta \nabla_\beta u^\alpha$$

$$\therefore \mathcal{L}_u \bar{\xi}^\alpha = u^\beta \nabla_\beta \bar{\xi}^\alpha - \bar{\xi}^\beta \nabla_\beta u^\alpha = u^\beta \nabla_\beta \xi^\alpha - \xi^\beta \nabla_\beta u^\alpha$$
$$= 0$$



$$\therefore \mathcal{L}_u \bar{\xi}^\alpha = 0$$

Now as t is arbitrary on cross curve ∴

∴ We can proceed with $\bar{\xi}^\alpha$ instead of ξ^α

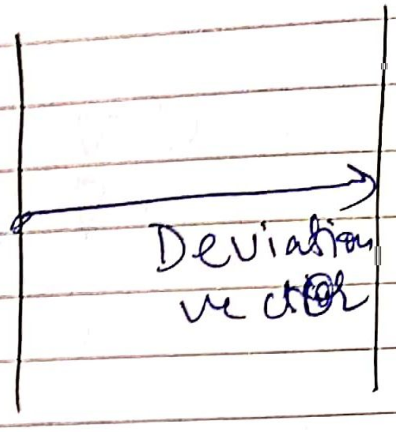
∴ We have freedom to set $\lambda = 0$

$$\therefore \xi^\alpha = \bar{\xi}^\alpha$$

∴ We can now impose

$$\boxed{u_\alpha \xi^\alpha = 0 \quad \text{on } \gamma_0}$$

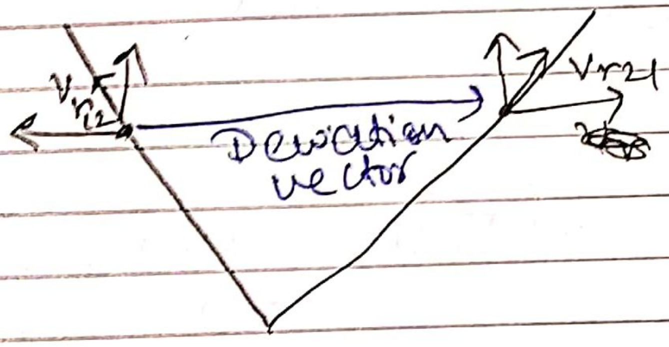
(12) If in a flat spacetime we are ~~st~~ having 2 straight lines with const. seperation then ∴ deviation is ~~constan~~ constant. time derivative & acc. of it = 0



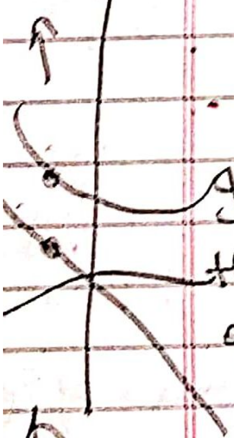
& in SR also acc. = 0

& in 3D By Galilean Relativity we can say so.

(13)



One is accelerated w.r.t. other at const rate in flat spacetime



If the lines diverge in flat spacetime their seperation ↑ But the relative velocity of one w.r.t. other is const.
∴ zero acc.

(14) ∴ In flat spacetime for Straight lines there is no Deviation acc.

Why Double Derivative? Bec. in SR acc = 0 ∴ check acc. in curved.

(15) $\frac{D}{dt}\frac{D}{dt}\xi^\alpha = \frac{D^2\xi^\alpha}{dt^2}$ = Relative acc. of γ_1 w.r.t. γ_0

$$\frac{D\xi^\alpha}{dt} = u^\beta \nabla_\beta \xi^\alpha$$

$$\frac{D^2\xi^\alpha}{dt^2} = u^\gamma \nabla_\gamma (u^\beta \nabla_\beta \xi^\alpha)$$

But we know

$$\mathcal{L}_u \xi^\alpha = u^\beta \nabla_\beta \xi^\alpha - \xi^\beta \nabla_\beta u^\alpha = 0$$

$$u^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta u^\alpha$$

$$\therefore \frac{D^2 \xi^\alpha}{dt^2} = u^\gamma \nabla_\gamma \left(\xi^\beta \nabla_\beta u^\alpha \right)$$

$$= u^\gamma \left((\nabla_\gamma \xi^\beta) \nabla_\beta u^\alpha + \xi^\beta \nabla_\gamma \nabla_\beta u^\alpha \right)$$

$$\nabla_\gamma \nabla_\beta u^\alpha - \nabla_\beta \nabla_\gamma u^\alpha = -R^\alpha_{\ \mu\gamma\beta} u^\mu$$

Instead we are going to put $u^i \xi_i = 0$ also

$$\frac{D^2 \xi^\alpha}{dt^2} = \left(\xi^\gamma \nabla_\gamma u^\beta \right) \nabla_\beta u^\alpha + u^\gamma \xi^\beta \left(\nabla_\beta \nabla_\gamma u^\alpha - R^\alpha_{\ \mu\gamma\beta} u^\mu \right)$$

$$= \xi^\gamma \nabla_\gamma u^\beta \nabla_\beta u^\alpha + \underbrace{u^\beta \xi^\gamma \nabla_\gamma \nabla_\beta u^\alpha}_{} - R^\alpha_{\ \mu\gamma\beta} u^\mu u^\gamma \xi^\beta$$

$$u^\beta \xi^\gamma \nabla_\gamma \nabla_\beta u^\alpha = \xi^\gamma \nabla_\gamma \left(\cancel{u^\beta \nabla_\beta u^\alpha} \right) - \xi^\gamma \nabla_\beta u^\alpha \nabla_\gamma u^\beta$$

We used the fact that they are geodesic lines.

for Geod. Eqn

$$\therefore \frac{D^2 \xi^\alpha}{dt^2} = -R^\alpha_{\ \mu\gamma\beta} u^\mu u^\gamma \xi^\beta$$

If $R \neq 0 \Rightarrow$ $\times$ Sym in any vectorfield $\Rightarrow \mathcal{L} g \neq 0$

If $R = 0 \Rightarrow$ Distance same $\Rightarrow$ Sym $\Rightarrow \mathcal{L} g = 0$?

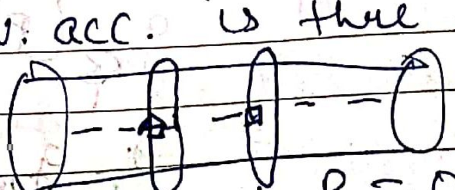
(16) Another way of thinking of flat spacetime

If $R = 0 \Rightarrow \frac{D^2 \xi^\alpha}{dt^2} = \text{Rel. Acc.} = 0$

$\Rightarrow$ By (14) Straight lines in flat spacetime.

(17) In sphere, Geod. Dev. acc. is there $\therefore R \neq 0$

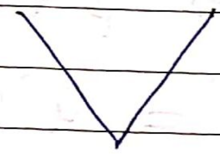
But in Cylinder



Non convergence, No Divergence $\therefore R = 0$

Wrapping up doesn't produce any intrinsic curvature.

Another way of thinking is that cylinder is just the wrap of plane paper & 2 straight lines on paper ~~will Remain~~ ~~∥ in cylinder~~ ..

(18) Now on a paper  $0 = \frac{D^2 \xi^\alpha}{dt^2}$

What if we now wrap paper to cylinder?

⇒ But as wrapping up doesn't produce any intrinsic curvature.

Then how to think in terms of converging of lines on cylinder?

Ans: They are conv./Div. on cylinder But at const Rate as in 2D plane

$\therefore \frac{D^2 \xi^\alpha}{dt^2} = 0$

(19) Here we are given 2 geodesics & we made many other.
But in congruence of geodesics we will be given many.

(20) Local flatness

∃ coordinate system such that at any point P in spacetime.

$$g_{\alpha\beta}(P) = \eta_{\alpha\beta}$$

$$\partial_\gamma g_{\alpha\beta}(P) = 0 \Rightarrow \Gamma^\alpha_{\beta\gamma}(P) = 0$$

Unless we are dealing with flat spacetime

$$\partial_{\alpha\beta}\, g_{ij} \neq 0$$

$\therefore$ By Expanding $g_{\alpha\beta}$

1st term is $\eta_{\alpha\beta}$, 2nd $= 0$, 3rd $= \partial_{\alpha\beta}\, g_{ij}$

But if $R = 0 \equiv$ flat Spacetime $\therefore$ $g_{\alpha\beta} = \eta_{\alpha\beta}$

All other terms $= 0$

(21) Rieman Normal Coordinates.

$\exists$ Coordinates x^α s.t. around P (at which $x^\alpha = 0$) origin

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} \underbrace{R_{\alpha\mu\beta\mu}}_{\boxed{\text{at } P}} x^\mu x^\mu + O(x^5)$$

at O : $x^\alpha = 0 \Rightarrow g_{\alpha\beta} = \eta_{\alpha\beta}$

at O : $\partial_i g_{\alpha\beta} \Rightarrow \partial_i g_{\alpha\beta} = 0$

$\therefore$ This particular Coordinate system & the particular metric produces Local flatness theorem.

(22) This also tells that
at O : $\partial_i \partial_j g_{\alpha\beta}$ are Related to $R_{\alpha\beta\gamma\mu}$.
And all nonzero $\partial_i\partial_j\, g_{\alpha\beta}$ form 20 Comp. of $R_{\alpha\beta\mu\gamma}$. (20 Components)

(23)

$$O(x^3) = \nabla_i R_{\alpha\mu\beta\gamma} x^\mu x^\nu x^i$$

(24) From Local flatness Th. we know
∃ coord. syst. s.t $g_{\alpha\beta} = \eta_{\alpha\beta}$, $\partial_i g_{\alpha\beta} = 0$, $\partial_i \partial_j g_{\alpha\beta} \neq 0$

But we did not get any info about coord. system.

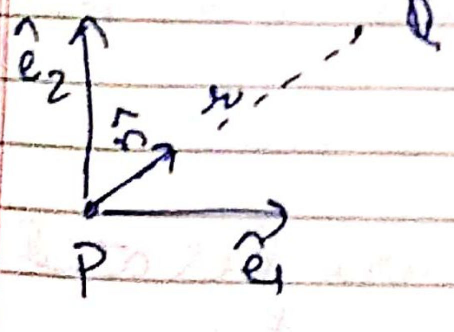
(25) from ~~Fermi~~ Riem. Normal coord.

We get info about the metric which will enforce Local flatness theorem

But
still no info on coordinate system.

(26) How to setup Cartesian coordinate in flat spacetime?

1. select Origin P
2. Pick frame / Basis vectors
3. Construct coordinates of Q
4. Unit vector along PQ
5. Decompose $\hat{n}$ into Basis

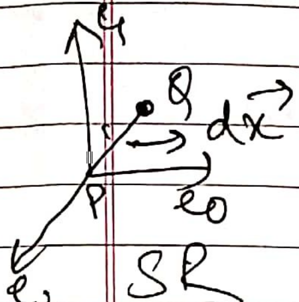


$$\hat{n} = n^j \hat{e}_j$$

Position vector of Q : $\vec{x} = r\hat{n}$
$\vec{x} = r n^j \hat{e}_j$

∴ Assign Q the coordinate
$x^j = r n^j$

(27) To prove metric $= \delta_{\alpha\beta}$ in 3D
metric $= \eta_{\alpha\beta}$ in 4D.



$\vec{dx} = dr\, n^j e_j \Rightarrow ds^2 = g(\vec{dx}, \vec{dx})$
$= dr^2 n^i n^j \delta_{ij}$

$ds^2 = M_{ij}\, dx^i dx^j \Rightarrow \delta_{ij} = M_{ij}$

SR

$ds^2 = M_{ij}\, dx^i dx^j = dr^2 n^i n^j \eta_{ij}$

? in small region

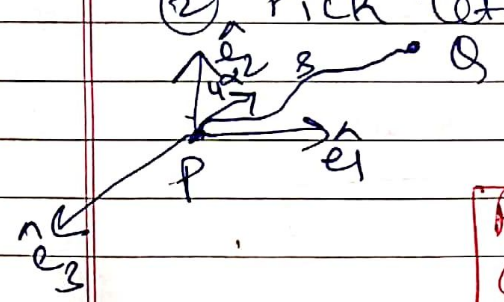
$M_{ij} = \eta_{ij}$

For GR also this goes on. But as in GR LIF: $g_{ij} \Rightarrow \eta_{ij}$ $\therefore$ verifying we got $M_{ij} = \eta_{ij}$

(28) In Manifold we have to change straight line from $P \to Q$ Bec. that was geod. in 2D.

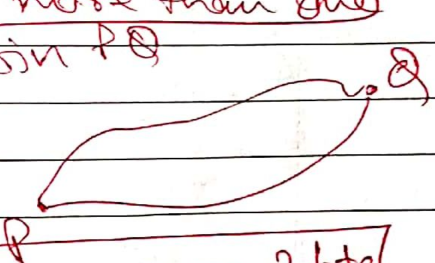
$\therefore$ Use geod. (Not straight) in curved spacetime

(1) Select Origin P
(2) Pick Tetrad / 4 Basis vector



(3) Draw Geodesic $P \to Q$

Assumption
Geod. $P \to Q$ has to be Unique
Only true if Q is near to P.
If Q is far away more than one geodesic can join PQ



(4) Proper Distance S
(5) Tangent vector U^α.

$U^\alpha = c^i e_i^\alpha$

$a^\alpha = \cancel{a}\, c^i e_i$

Now we can't do this
→ Step i.e. we can define $\vec{x}$ PQ on manifold

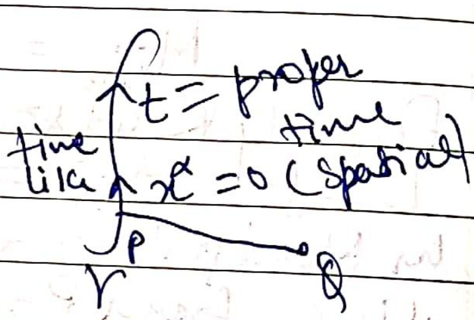
But on sphere any 2 pts can be connected with non Unique Geod.

b)

We can directly go to $x^{\gamma} = s\, n^{\gamma}$
By Declaring RNC : $x^{\gamma} = s\, C^{0\gamma}$
Metric we get will be (21)

(29)

(30) Fermi Normal Coordinates



Instead of taking point as Origin, Take like geod. as spatial Origin. To get spatial coord. of Q s.t. PQ is $\perp$

L-5

① Geodesic Congruences

It can be a geodesic or not.

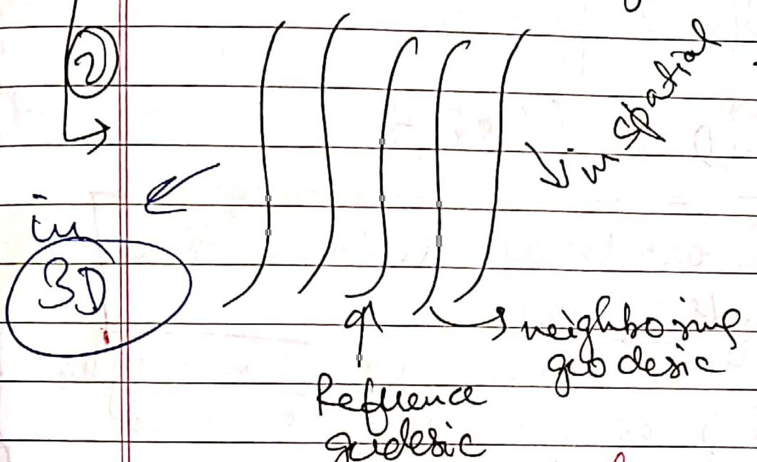
Congruence : Family of curves that don't intersect.

∴ Only one curve passes through each event.

If the geodesics intersect : we say singularity of congruence

But we will ignore this.

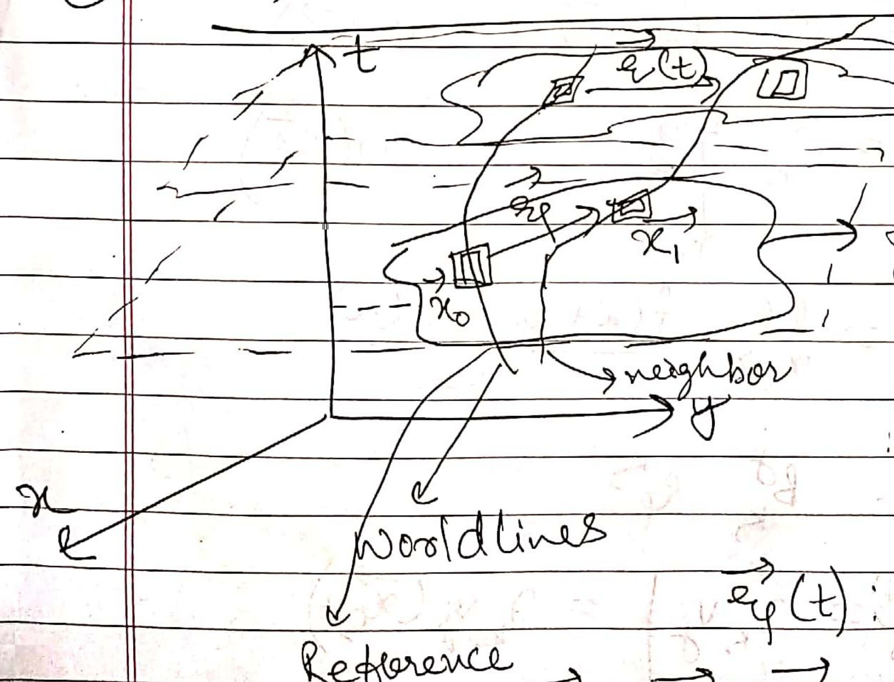
②



To describe neighboring geodesic w.r.t Reference geodesic ?

fluid element is small on macroscopic scale but in microscopic scale has large no. of molecules & each fluid elem. has its pressure, & other properties

③ 3D Newtonian Fluid Mechanics



→ fluid configuration made up of many fluid elements. & each fluid element has its own properties.

$\vec{\xi}(t)$: Displacement b/w given fluid element & Reference fluid element

$\vec{\xi} = \vec{x}_1 - \vec{x}_0$

The worldline of fluid elements can be thought of as congruence.

in manifold $\vec{\xi}_1 = \left(\frac{\partial x^\mu}{\partial s}\right)$

(4) in flat spacetime

$$\vec{\xi}_1(t) = \vec{x}_1 - \vec{x}_0$$

We can define $\vec{\xi}_1(t)$ as diff. of other two vectors

But
in curved spacetime
Diff. of 2 vectors is not a vector
$\therefore$ we can't define $\vec{\xi}_1(t)$ as above.

(5) Relative velocity B/w 2 particles

$$\frac{d\vec{\xi}_1}{dt} = \vec{V}(\vec{x}_1, t) - \vec{V}(\vec{x}_0, t)$$

If $\vec{x}_1$ & $\vec{x}_0$ are at larger distance then we can't do anything else

But if $\vec{x}_0$ & $\vec{x}_1$ are close we can taylor expand $V(\vec{x}_1, t)$ w.r.t. x_0 as x_0 is our reference.

(6) In component form

$$\frac{d\xi_1^j}{dt} = v^j(\vec{x}_0 + \vec{\xi}_1) - v^j(\vec{x}_0)$$

$$= \partial_k v^j \Big|_{x_0} \xi_1^k + O(\xi_1^2)$$

as $\frac{\partial f}{\partial x} = \lim_{x_0 \to 0} \frac{f(x+x_0) - f(x)}{x_0}$.

$$\therefore \frac{d\xi_1^j}{dt} = B^j_k \xi_1^k$$

$$B_{jk}(t) = \partial_k v_j \Big|_{x_0} = \partial_k v_j(\vec{x}_0, t)$$

Th: Every Symmetric Matrix can be decomposed into Trace part & Trace free part.

Proof: Let $Tr\, S = \theta$

$\therefore \sigma_{11} + \sigma_{22} + \sigma_{33} = \theta$

For each $\sigma_{11} = x_{11} + \theta/3$

$\sigma_{22} = x_{22} + \theta/3$

$\sigma_{33} = -(x_{11}+x_{22}) + \theta/3$

$\therefore$ ~~Tr. free~~ Trace part $= \begin{bmatrix} \theta/3 & & 0 \\ & \theta/3 & \\ 0 & & \theta/3 \end{bmatrix}$

Trace free part $= \begin{bmatrix} x_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & x_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & -(x_{11}+x_{22}) \end{bmatrix}$

Date ______
Page ______ 11

(7) B^j_k is a 3×3 matrix
But B^j_k has no symmetry.

∴ Break down it into smaller pieces (irreducible forms)

∴ Decompose it into irreducible pieces.

$$B_{jk} = \underbrace{\underset{\text{Tracepart}}{\frac{\delta_{jk}}{3}\theta} + \underset{\text{Sym. Trace free}}{\sigma_{jk}}}_{\text{Sym.}} + \underset{\text{A.S.}}{\omega_{jk}}$$

where $\theta = \delta^{jk} B_{jk}$ as $\delta^{jk} B_{jk} = \frac{\delta^{jk}\delta_{jk}}{3}\theta$

$$\delta^{jk} B_{jk} = \theta$$

(8) $B_{(jk)} \equiv$ Sym. of B_{jk}

$B_{[jk]} =$ Antisy of B_{jk}

How can I assume Diag. would all be same?

$\therefore \omega_{jk} = B_{[jk]}$

$$\sigma_{jk} = B_{(jk)} - \frac{\delta_{jk}}{3}\theta$$

(9) $$\frac{\delta_{jk}}{3}\theta = \begin{pmatrix} \frac{\theta}{3} & 0 & 0 \\ 0 & \frac{\theta}{3} & 0 \\ 0 & 0 & \frac{\theta}{3} \end{pmatrix} \equiv \text{Expansion tensor}$$

b)

$$\sigma_{jk} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & -(\sigma_{11}+\sigma_{22}) \end{pmatrix} \equiv \text{Shear Tensor}$$

$$\omega_{jk} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} = \text{Rotation Tensor}$$

(10) σ_{jk} has 5 components

$B_{(jk)} = {}^4C_2 = 6$

But σ_{jk} components $\Rightarrow \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}$

$\frac{\delta_{jk}\theta}{3} \Rightarrow$ 1 component

$\rightarrow$ 6 Total

ω_{jk} has 3 comp.

$\therefore$ Total 9 components which are components of B_{jk}.

(11) What is the significance of each piece on fluid?

$\rightarrow$ Trace of B_{jk}

(12) $\theta = \delta^{kj} B_{jk} = \delta^{kj} \partial_k V_j = \vec{\nabla} \cdot \vec{V}$

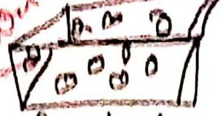
Expansion

= Divergence of vector field
= Rate of expansion of fluid element as they move around fluid

How?

(13) Take one fluid element which has same no. of molecules
$\therefore$ Remain fixed $\Rightarrow$ Mass conservation

Assumption



fluid element $\Rightarrow$ ω molecules remain fixed $\Rightarrow$ mass remains fixed in fluid element

$\Rightarrow$ But volume of fluid element can change.

But volume can change.

Mass conservation is described by continuity Eq^n

Continuity Eq^n $\quad \partial_i J^i = 0 \qquad J^i \equiv (\rho, \rho\vec{v})$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Here let
$$\boxed{\begin{aligned} \rho &= \frac{\delta m}{\delta V} = \text{mass Density} \\ \rho\vec{v} &= \text{mass Current Density} \end{aligned}}$$
$\delta m \rightarrow$ mass in the fluid element
$\delta V \rightarrow$ fluid element volume

$\therefore$ Continuity Eq^n

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

(14) $\quad \vec{\nabla}\cdot(f\vec{A}) = \frac{\partial(fA_1)}{\partial x} + \frac{\partial(fA_2)}{\partial y}$

$$= f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f.$$

$$\therefore \frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = -\rho\,\vec{\nabla}\cdot\vec{v}$$

$$\left\{ \frac{d\rho}{dt} = -\rho\,\vec{\nabla}\cdot\vec{v} \right.$$

density of the same fluid element

$\boxed{\text{As we follow the given fluid element}}$

at t : $\rho(t, \vec{x})$

at $t+dt$: $\rho(t+dt, \vec{x}+d\vec{x})$

$d\rho = \rho(t+dt, \vec{x}+d\vec{x}) - \rho(t, \vec{x})$

$= \cancel{\rho(t) + \frac{\partial \rho}{\partial t}dt + \rho(x)} = \frac{\partial \rho}{\partial t}dt + (\vec{\nabla}\rho)\cdot d\vec{x}$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{\nabla}.\rho)\vec{v} = \text{Convective derivative}$$

→ If I move along the fluid then $\frac{d\rho}{dt}$

But what is the time derivative of ρ always at one point then $d\rho = \rho(t+dt, x) - \rho(t, x)$

$d\rho = \frac{\partial \rho}{\partial t} dt \Rightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t}$

Density Different fluid elem

(15) Convective derivative in 4D notation

→ $\underbrace{\frac{\partial \rho}{\partial t} + (\nabla \rho)\cdot\vec{v}}_{\text{in Nonrel.}} = u^\alpha \partial_\alpha \rho$

As $u^\alpha \partial_\alpha \rho = u^0 \frac{\partial \rho}{\partial t} + u^1 \frac{\partial \rho}{\partial x^1}$

$$= \frac{dt}{d\tau}\frac{\partial \rho}{\partial t} + \frac{dx}{d\tau}\frac{\partial \rho}{\partial x^1}$$

$d\tau = \frac{dt}{\gamma}$

$$= \gamma \frac{\partial \rho}{\partial t} + \gamma v_x \frac{\partial \rho}{\partial x}$$

$$\approx \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x}$$

Import

(16) Therefore see $u^\alpha \partial_\alpha \rho$ as something which keeps tracks of how ρ is changing along the world line.

But in GR $u^\alpha \partial_\alpha \rho \longrightarrow u^\alpha \nabla_\alpha \rho = \frac{D\rho}{dt}$ ← Covariant derivative

Imp

$= \frac{d\rho}{dt}$

Geod. eq $u^\alpha \nabla_\alpha u^\beta = 0$

∴ How u^β is changing along the world line

$\therefore \underline{u^\alpha \partial_\alpha \rho} = -\rho \nabla \cdot v$

$\hookrightarrow$ how ρ changes along the worldline.

(17) Now from (14)

$$-\frac{1}{\rho}\frac{d\rho}{dt} = \vec{\nabla}\cdot\vec{v}$$

But $\rho = \dfrac{\delta m}{\delta V}$ if m is conserved.

then $\dfrac{d}{dt}\left(\dfrac{\delta m}{\delta V}\right) = -\dfrac{\delta m}{(\delta V)^2}\dfrac{d(\delta V)}{dt}$

$$\therefore \vec{\nabla}\cdot\vec{v} = \frac{1}{(\delta V)}\frac{d}{dt}(\delta V)$$

$\therefore$ Divergence of velocity field depends on Volume of fluid element

If Div. is +ve then fluid element expands in time

$\therefore$ Expansion element is fractional Rate of change of volume of fluid element.

(18) By (12) $\theta = \dfrac{1}{\delta V}\dfrac{d}{dt}(\delta V)$ = Rate of Expansion of fluid element as they move along fluid.

θ Expansion Parameter

(19) $\underline{\text{Shear}}$

$\underline{\text{let}}$ $\theta = \omega_{jk} = 0$

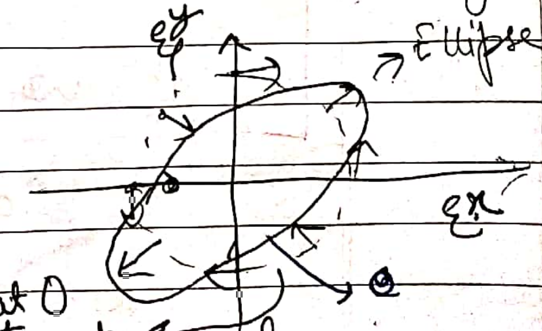
$\therefore \dfrac{d\xi^j}{dt} = \sigma^j{}_k\, \xi^k$ $\rightarrow$ 5 components

How displacement vectors change

let $\sigma^j{}_k = \sigma^1{}_2$ only $\uparrow$ All others vanishing

$$\therefore \frac{d\xi^x}{dt} = \sigma\, \xi^y$$

$$\frac{d\xi^y}{dt} = \sigma\, \xi^x$$



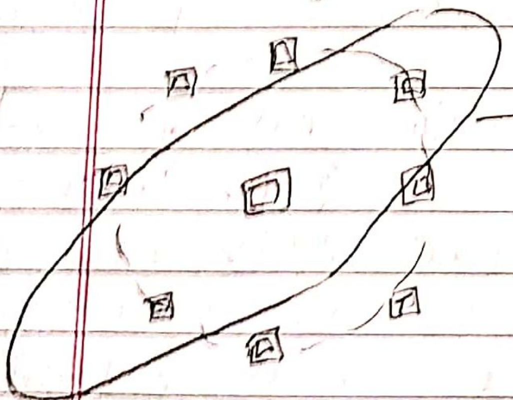
Assuming Ref. element at O & which fluid element makes circle

Shear Tensors makes the neighboring fluid elements squeezed from initial shape to ellipse.

To prove volume will Remain preserved

(20) Shear Tensor

∴ There is deformation of shape without change in volume.



→ Shape changes
Volume same.

What if we also take θ along with σ ?

(21) For all σ^j_k we will Repeat the same. & we get sphere changes its shape to ellipsoid without changing volume

(22) Rotation

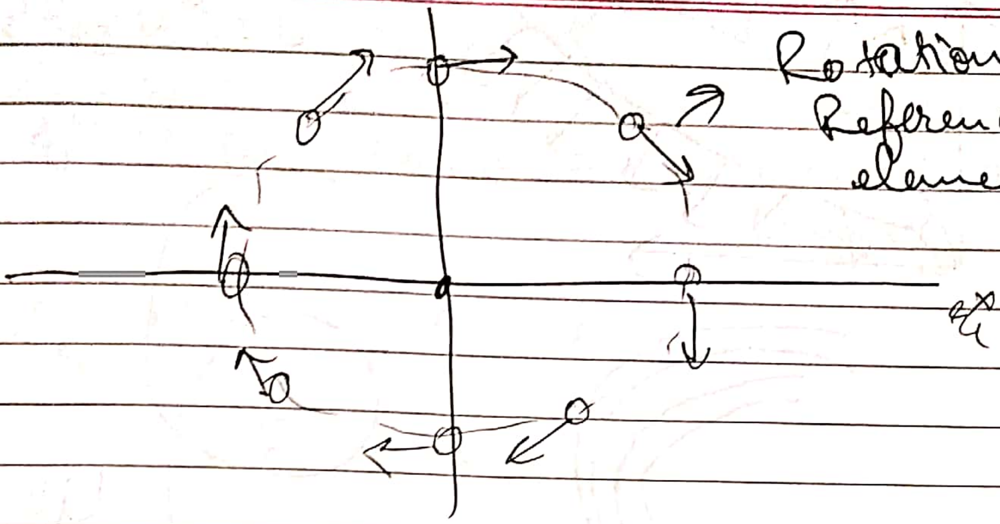
Let $\theta = \sigma_{jk} = 0$

Let $\omega_{12} = \omega \neq 0$ All others vanishing

$$\therefore \frac{d e^x_\varepsilon}{dt} = \omega e^y_\varepsilon$$

$$\frac{d e^y_\varepsilon}{dt} = -\omega e^x_\varepsilon$$

eg.



Rotation w.r.t. Reference fluid element.

Shape Remains Same, volume Remains Same,
just there is a Rotation, without distortion

ω = Rate of Rotation = Angular velocity.

(23) Taking others ω as non vanishing we get spherical Ball Rotating in arbitrary direction.

(24) The starting point of all this was

$$\frac{d\xi^j}{dt} = v^j(\vec{x}_0+\vec{\xi}) - v^j(\vec{x}_0)$$

$$= \partial_k v^j(x_0)\,\xi^k + O(\xi^2)$$

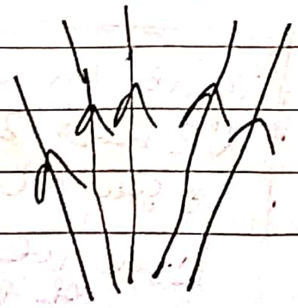
Linear Transf^n

$\boxed{\text{what if we take these terms?}}$

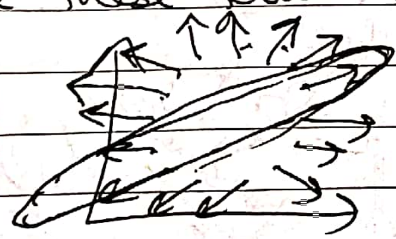
In Old Elasticity books they do take these terms & solve.

But in GR we don't assume both Geod. to be close

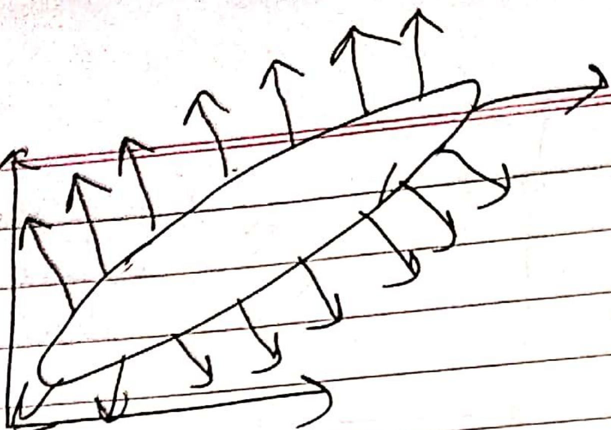
(25) By θ



By σ



$\vec{\nabla}\cdot\vec{v} = \frac{1}{\delta V}\frac{d(\delta V)}{dt}$



By ω

Like Helix

(26) Previous arguments didn't take use of fluid elements but just of Congruence

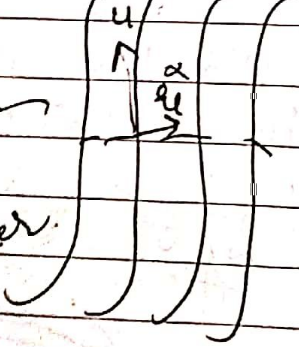
Only place where we use of fluid element is mass conservation. θ, ω, σ ← ∴ Everything in GR is same as previous

One could derive $\theta = \frac{1}{\delta V} \frac{d}{dt}(\delta V)$

This is not essentially tied to mass cons. & can be derived for gen. Cong-Curves

(27) How to go to Non-Newtonian fluid?

(28) Congruences : Family of Curves that don't intersect
|||
Only one curve passing through each event.



Time like Geodesic

C: Proper time parameter.

Earlier in Geod. Deviation: We had 2 curves & we postulated curves in B/W

Now we have family of non-int. curves

$\Rightarrow u^\beta \nabla_\beta u^\alpha = 0 \quad ; \quad u^\alpha u_\alpha = 1$

$\Rightarrow \xi_\varphi^\alpha$: Deviation vector = Tangent to Cross Curves.

$\Rightarrow \mathcal{L}_{\xi_\varphi} u^\alpha = \mathcal{L}_u \xi_\varphi^\alpha = 0$

$$u^\beta \nabla_\beta \xi_\varphi^\alpha = \xi_\varphi^\beta \nabla_\beta u^\alpha$$

$\Rightarrow \xi_\varphi^\alpha u_\alpha = 0$

(29) Decomposition of g_{ab} into time dir & spatial.

$$g_{ab} = \underbrace{u_a u_b}_{\text{time}} + \underbrace{h_{ab}}_{\text{spatial}}$$

$\cancel{h_{ab} u^b = 0}$

$$h_{ab} = g_{ab} - u_a u_b$$

(30) $h_{ab} u^b = u_a - u_a = 0$

Right

$$\boxed{\therefore u^b h_{ab} = 0}$$

Left $u^a h_{ab} = u_b - u_b = 0$

$\therefore h_{ab}$ Represents spatial displacement

See (37)

(31) h_{ab} = Transverse metric
g_{ab} = Longitudinal metric

(32) I have preffered vector field in spacetime which is tangent vector field to congruence which can be used to define preffered time direction. Put myself in a frame in which vector field is oriented along time. That frame is generally (UR) comoving frame

(33) Metric which represents spatial Displacement $= h_{ab}$

as h_{ab} is Orth. to Time direction which is the direction of Tang. vectors.

Properties

$h^{ab} = g^{ab} - u^a u^b$

(34) ① $h^a_i h^i_b = h^a_b$

$h^a_i h^i_b = (\delta^a_i - u^a u_i)(\delta^i_b - u^i u_b)$

Proof $= \delta^a_b - u^a u_b - u^a u_b + u^a u_b$

$= h^a_b$

② $u^a h_{ab} = 0$

③ $h^i_i = +3$

$Tr(h^a_b) = Tr(g^a_b) - Tr(u^a u_b)$

$h^i_i = \frac{g}{d} - u^a u_a = 4 - 1 = 3$

(35) Projection Operator

$\Rightarrow h_{ab}$ is the Projection Operator. How? Prove?

All Projection operators have

$\delta^a_b = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$g_{ab} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$h^a_i h^i_b = h^a_b$ property

& as $h_{ab} u^b = 0$

$\therefore h_{ab}$ is Proj. Op.

h_{ab} projects to the spatial part

$\Rightarrow T^{ab} = (p+\rho) u^a u^b - p g^{ab}$

$T^a_b = (p+\rho) u^a u_b - p g^a_b = \rho u^a u_b + p(-g^a_b + u^a u_b)$

$= \rho u^a u_b - h^a_b p$

But h^a_b is the projection operator.

in Rest frame

h_{ab} is the spatial part

$\therefore p$ acts on Space only

why $g_{ab} = U_a U_b + h_{ab}$?

Date
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(34) In a local Lorentz frame momentarily comoving with referance geodesic.

$\rightarrow u^\alpha \stackrel{*}{=} (1, 0, 0, 0)$

$$g_{\alpha\beta} \stackrel{*}{=} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Only for Spacelike surface

~~what about Timelike surface?~~

$\rightarrow$ How of trace of $g^i_j = 4$?

we want $h_{\alpha\beta} \stackrel{*}{=} \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Spatial metric

which we can get by

$$h_{\alpha\beta} \stackrel{*}{=} g_{\alpha\beta} - u_\alpha u_\beta$$

But It is Tensorial eqn $\therefore$ Valid in any frame

$$\therefore h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$$

(35) Behaviour of Neighbouring geodesic relative to Ref. geod. is given by

$$U^\beta \nabla_\beta \xi^\alpha = \nabla_\beta U^\alpha \xi^\beta \equiv U^\beta \nabla_\beta \xi^\alpha = B^\alpha_\beta \xi^\beta$$

Finally $\frac{d\xi^k}{dt} = \partial_j v^k \xi^j$ (It was an approx.)

$$\therefore \boxed{B_{\alpha\beta} = \nabla_\beta U_\alpha} \Rightarrow \text{Gradient velocity}$$

39 In fluid case the eqn was an approx. of Taylor series

Imp

Here we have not used any of it.
It is Exact.

This is exact Bcz we have used $\vec{\xi}$ Deviation vector as Tangent to cross curves.

40 In fluid case No time component came as we were working in 3D
Here also
Although working in 4D
ξ^α has no time component
as $u_\alpha \xi^\alpha = 0$ By 28

41 $\boxed{B_{\alpha\beta} u^\beta = u^\alpha B_{\alpha\beta} = 0}$ Proof

$\therefore B_{\alpha\beta}$: Purely spatial

$\boxed{\text{But } B_{\alpha\beta} \text{ has not symmetry}}$

(1) $B_{\alpha\beta} u^\beta = \underbrace{u^\beta \nabla_\beta u_\alpha = 0}$

Imp ← Only True for Timelike Affine Geod.
$B^{\alpha\beta}$ is not spatial ← for Non Geod. Not true

(2) $u^\alpha \nabla_\beta u_\alpha = \nabla_\beta u^\alpha u_\alpha$

$= \dfrac{u^\alpha \nabla_\beta u_\alpha + u_\alpha \nabla_\beta u^\alpha}{2} \Rightarrow \dfrac{\nabla_\beta (u_\alpha u^\alpha)}{2} = 0$

Imp Only if Time like Geod. is Used

(42) As in fluid also $B_{\alpha\beta}$ was purely spatial
∴ Same Decompⁿ as in fluid

(43)

valid even if it is not Geodesic

Imp Not $g_{\alpha\beta}$ as $B_{\alpha\beta}$ is purely spatial ∴ use $h_{\alpha\beta}$

(44) Decomposition of $B_{\alpha\beta}$

$$B_{\alpha\beta} = \frac{h_{\alpha\beta}}{3}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

Purely Spatial ∴ No time comp. ∴ use $h_{\alpha\beta}$

If in local comoving frame no time compo. of $\sigma_{\alpha\beta}, \omega_{\alpha\beta}, h_{\alpha\beta}$

Proof ∴ All these Tensor Orth. to u^α
let $\Theta, \omega_{\alpha\beta} = 0$ ∴ $B_{\alpha\beta} = \sigma_{\alpha\beta}$
$\Rightarrow u^\alpha B_{\alpha\beta} = 0 = u^\alpha \sigma_{\alpha\beta}$ ∴ Spatial

(45) $\Theta = h^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta}\nabla_\beta u_\alpha$

How? Tr Θ?

as $h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ $= \nabla_\beta u^\beta$

Covariant Divergen.

Imp $h_{\alpha\beta}\sigma^{\alpha\beta} = 0$ → Equivalent to $g^{\alpha\beta} A_{\alpha\beta}$
$h^{\alpha\beta}\omega_{\alpha\beta} = 0$

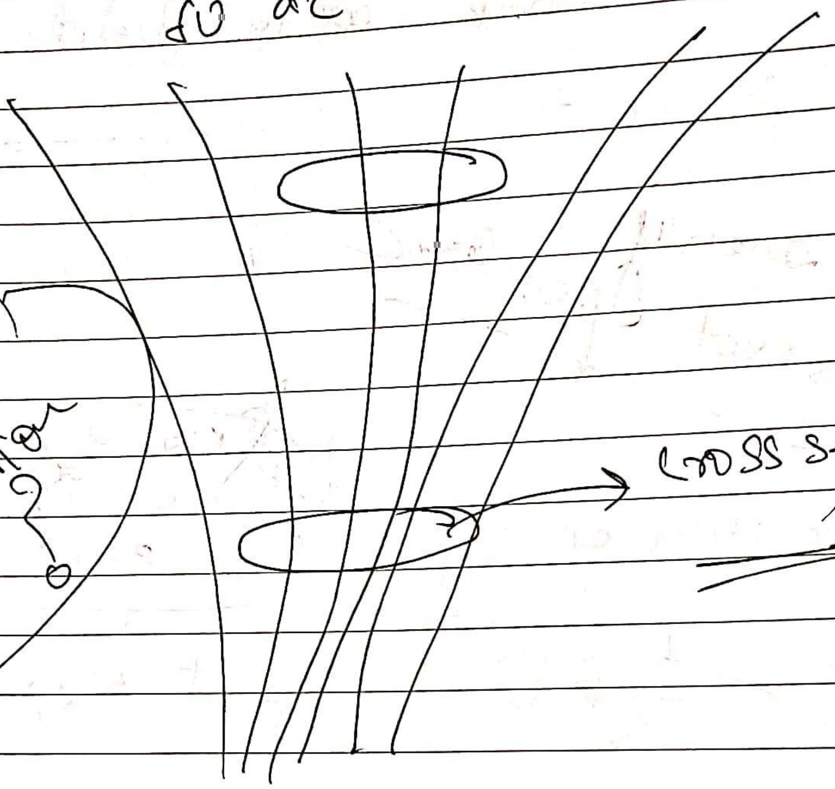
Tr$(A_{\alpha\beta})$ Analogy $\vec{\nabla}\cdot\vec{v}$

S.A? ⇒ ∴ Tr $\sigma = 0$

$$\sigma_{ab} = B_{(ab)} - \frac{h_{ab}\Theta}{3}$$

$$\omega_{ab} = B_{[ab]}$$

(46) $\Theta = \frac{1}{\delta V} \frac{d}{d\tau} (\delta V)$



cross sectional volume

Physical interpretation of Θ ?

(47) All the parameters analogy same as fluid.

(48) $h_{\alpha\beta}$ is Sym

$h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$

$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \Rightarrow$ $g_{\alpha\beta}$ Sym.

$U_\alpha U_\beta = U_\beta U_\alpha$

what about Arbitrary $A^\alpha A^\beta = A^\beta A^\alpha$?

HW1 1.13 ⇒ 3, 6, 9
2.5 ⇒ 1, 3

Date 17 Aug

L-6

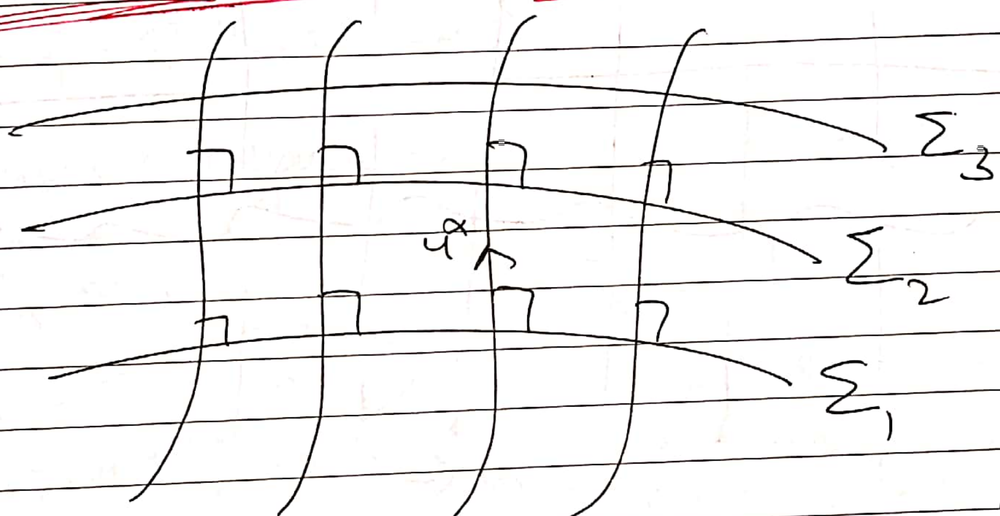
① What is the meaning of θ in terms of Geodesic Curves?

② Frobenius Theorem → Geodesic / Non Geodesic

Congruence of ~~timelike Geod~~. is Hypersurface Orthog. iff $\omega_{\alpha\beta} = 0$ for Geod curve

in general for all curves $u_{[\alpha}u_{\beta;\gamma]} = 0$

i.e. We have family of Hypersurfaces s.t. normal vector is everywhere aligned with u^α.



(Not give full proof as it Requires Forms)

③ What is Hyper Surface?

It is a Submanifold of a spacetime manifold that has fewer Dim than it.

④ For Timelike Geodesics

Σ are spatial Surfaces

if Curve is spatial in one pt it is spatial all along.

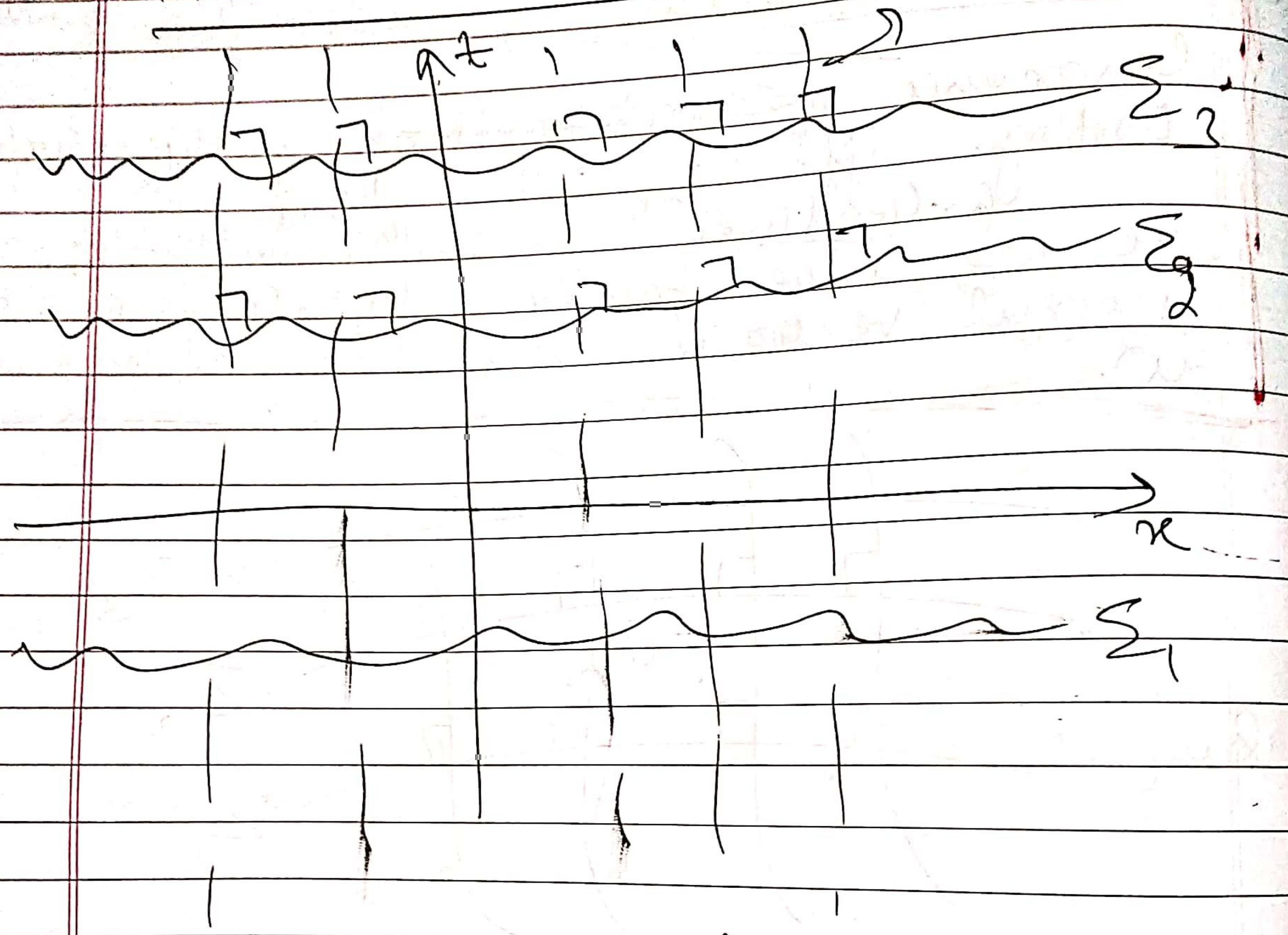
as in LIF comoving $u^\alpha \stackrel{*}{=} (1, 0, 0, 0)$

∴ as Σ are $\perp^r$ to u^α ∴ Σ are Spatial

Tensorial

⑤ Example

Flat Minkowski Space



Σ are Orthogonal to u^i = Tensorial
But $u^i = (1,0,0,0)$

$\therefore$ Σ is spatial.

⑥ Hyper Surface $\quad \phi(x^\alpha) = \text{const}$

2 Sphere in 3D $\rightarrow$ 2 Dim

$\phi(x,y,z) = x^2+y^2+z^2 = R^2$

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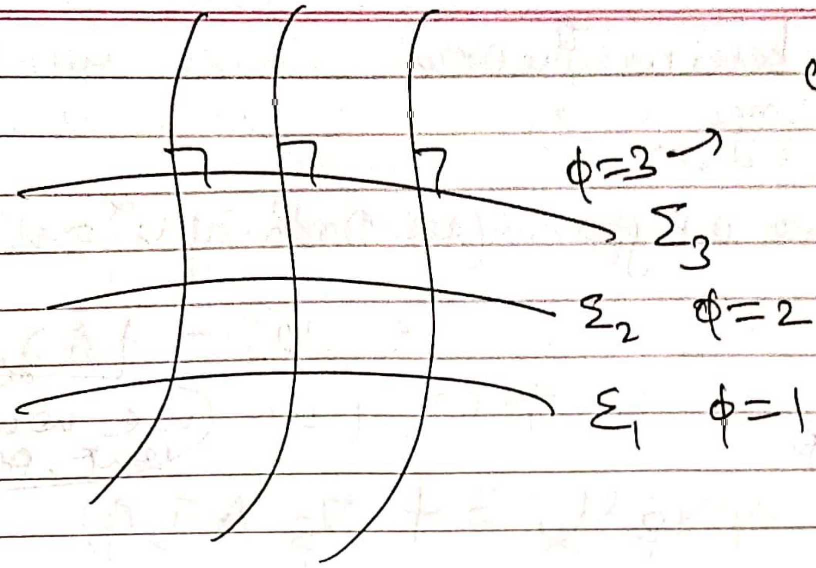

constants would tell which hypersurface I am talking about.

(7) Normal to the surface is define as Grad. of that fn.

Normal to Hypersurface $n_\alpha \propto \partial_\alpha \phi$

1 form

$\propto$ so as to Normalize.

To make n^α future pointing

Why -ve? in $-+++$ Notation

$n_\alpha = -A\,\partial_\alpha \phi$

A is the Scalar fn for Normalization.

$$\boxed{n_\alpha n^\alpha = -1}$$

Bec. $\partial_\alpha \phi$ is $\uparrow$ as we go up as $\phi \uparrow$ But to keep n_α timelike $\therefore$ put -ve
$n_\alpha = -A\,\partial_\alpha \phi$
& $n_\alpha n^\alpha = -1$

as $n^0 n_0$
as $n_0 = -n^0$
$\therefore n^0 = -n_0$

In Our Notation

$$\boxed{\begin{gathered} n_\alpha = A\,\partial_\alpha \phi \\ n_\alpha n^\alpha = 1 \end{gathered}}$$

What if we suppose $\phi \downarrow$ then?

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(8) $\therefore$ Frobenius Theorem now says.

Congruence is Hypersurface Orth. if $u^\alpha = n^\alpha$

i.e. $u_\alpha = + A \partial_\alpha \phi$

As $B_{\alpha\beta} = \theta + \sigma + \omega$ (we want to look on ω part)

(9) $\therefore B_{\alpha\beta} = \nabla_\beta u_\alpha = + \nabla_\beta (A \nabla_\alpha \phi)$ as ϕ, A are scalar fn

$= + A \nabla_\beta \nabla_\alpha \phi + (\nabla_\alpha \phi) \nabla_\beta A$

$= + A \nabla_\beta \nabla_\alpha \phi + \partial_\alpha \phi \partial_\beta A$

$= + A \partial_\beta \partial_\alpha \phi + \partial_\alpha \phi \partial_\beta A$ as A & ϕ are scalar fn

Doubt: df is also scalar if f is scalar $\therefore \nabla^2 f = d^2 f$

As 1st part is symmetric

$$\partial_\beta \partial_\alpha \phi = \underbrace{\frac{\partial_\beta \partial_\alpha \phi + \partial_\alpha \partial_\beta \phi}{2}}_{\text{Sym}} + \underbrace{\cancel{\frac{\partial_\beta \partial_\alpha \phi - \partial_\alpha \partial_\beta \phi}{2}}}_{\text{A}}$$

$$\partial_\beta \partial_\alpha \phi = \frac{\partial_\beta \partial_\alpha \phi + \partial_\alpha \partial_\beta \phi}{2} = \underline{\text{Sym}}$$

$\therefore$ All Antisym will be from $\partial_\alpha \phi \partial_\beta A$

~~$\omega_{\alpha\beta} = -\partial_\alpha \phi \partial_\beta A$~~

$$\therefore \omega_{\alpha\beta} = + \frac{1}{2} \left(\partial_\alpha \phi \partial_\beta A - \partial_\beta \phi \partial_\alpha A \right)$$

$$= \frac{1}{2A} \left(u_\alpha \partial_\beta A - u_\beta \partial_\alpha A \right)$$

(10)

(11)

(10) As we know from previous

Imp $\omega_{\alpha\beta} U^{\beta} = 0$ as $\omega_{\alpha\beta}$ is spatial see (44)

$\omega_{\alpha\beta}$ is spatial Tensor like $\sigma_{\alpha\beta}$, $B_{\alpha\beta}$

$$\therefore \quad 0 = U^{\alpha}\omega_{\alpha\beta} = \frac{1}{2A}\left(\partial_{\beta}A - U^{\alpha}U_{\beta}\partial_{\alpha}A\right)$$

$$\therefore \quad \partial_{\beta}A = (U^{\alpha}\partial_{\alpha}A)\,U_{\beta}$$ (Only for timelike Geod)

Imp $\boxed{\text{A can vary But Only in Direction of } U_{\beta}\text{. A must be constant on each Hypersurface.}}$

(11) But putting it in (9)

$$\omega_{\alpha\beta} = \frac{1}{2A}\left(U_{\alpha}\left(U^{i}\partial_{i}A\right)U_{\beta} - U_{\beta}\left(U^{i}\partial_{i}A\right)U_{\alpha}\right)$$

$$= \frac{1}{2A}\left(\cancel{U_i U^{\alpha}\partial_{\alpha}A}\,(U^{i}\partial_{i}A)\left[U_{\alpha}U_{\beta} - U_{\beta}U_{\alpha}\right]\right)$$

But

$U_{\beta}U_{\alpha} = U_{\alpha\beta}^{U}$

$$\boxed{\partial_{\alpha}A\,\partial_{\beta}\phi = \partial_{\beta}A\,\partial_{\alpha}\phi}$$ (56)

$$\therefore \quad \omega_{\alpha\beta} = 0$$

$dx\,dy = dy\,dx$

as $g_{\alpha\beta}\underbrace{dx^{\alpha}dx^{\beta}}_{\text{Sym}} = ds^2$

$\therefore$ $g_{\alpha\beta}$ Sym.

$$U^{\alpha}U^{\beta} = \frac{dx^{\alpha}dx^{\beta}}{d\tau^2}$$ → Sym.

$\therefore$ $U^{\alpha}U^{\beta}$ Sym

(12) Proof Inverse Also holds

if $\omega_{\alpha\beta} = 0 \Rightarrow U_\alpha = -A\,\partial_\alpha \phi$

(13) $\omega_{\alpha\beta} = 0 \equiv U_{[\alpha} U_{\beta,\nu]} = 0$

for timelike Geod Par τ

for any

Proof:

(14) Intuitive meaning of why $\omega_{\alpha\beta} = 0$?

(15) Till now all we have done is kinematics, We did not use Euler Lagrange Eqn of fluid Elements?
We are checking what would be Rel. acc. of Neighboring geodesics be?
But we are not caring about what is causing acceleration?
in Geod. Dev. we were doing Dynamics?

(16) Evolution eqn of $B_{\alpha\beta}$:

$$\frac{D\xi^\alpha}{dt} = \xi^\beta \nabla_\beta u^\alpha = u^\beta \nabla_\beta \xi^\alpha$$

$$\frac{D\xi^\alpha}{dt} = \xi^\beta \nabla_\beta u^\alpha$$

$$\frac{D^2\xi^\alpha}{dt^2} = \frac{D}{dt}\left(\xi^\beta \nabla_\beta u^\alpha\right)$$

$$= u^\gamma \nabla_\gamma\left(\xi^\beta \nabla_\beta u^\alpha\right)$$

$$= u^\gamma \xi^\beta \nabla_\gamma \nabla_\beta u^\alpha + u^\gamma \nabla_\beta u^\alpha \nabla_\gamma \xi^\beta$$

$$u^\gamma \xi^\mu R^\alpha_{\ \beta\gamma\mu} u^\beta \qquad = u^\gamma \xi^\beta \nabla_\gamma \nabla_\beta u^\alpha + \nabla_\beta u^\alpha \xi^\gamma \nabla_\gamma u^\beta$$

$$= u^\gamma \xi^\beta \left(R^\alpha_{\ \mu\gamma\beta} u^\mu + \nabla_\beta \nabla_\gamma u^\alpha\right) + (\nabla_\beta u^\alpha)\xi^\gamma \nabla_\gamma u^\beta$$

$$= u^\gamma \xi^\beta R^\alpha_{\ \mu\gamma\beta} u^\mu + \xi^\beta \nabla_\beta\left(u^\gamma \nabla_\gamma u^\alpha\right) - \xi^\beta \nabla_\beta u^\gamma \nabla_\gamma u^\alpha + (\nabla_\beta u^\alpha)\xi^\gamma \nabla_\gamma u^\beta$$

$$\therefore \left(\nabla_\beta u^\alpha\right) \xi^\gamma_\varphi \nabla_\gamma u^\beta = \xi^\beta_\varphi \nabla_\gamma u^\alpha \nabla_\beta u^\gamma$$

$$B^\alpha_\beta \, \xi^\gamma_\varphi \, B^\beta_\gamma = \xi^\beta_\varphi \, B^\alpha_\gamma \, B^\gamma_\beta$$

$$B^\alpha_\gamma \, \xi^\beta_\varphi \, B^\gamma_\beta = \xi^\beta_\varphi \, B^\alpha_\gamma \, B^\gamma_\beta$$

$$\boxed{B^\alpha_\gamma \, \xi^\beta_\varphi = \xi^\beta_\varphi \, B^\alpha_\gamma}$$

(17) $$\frac{DB_{\alpha\beta}}{d\tau} = \nabla_\mu B_{\alpha\beta} \, u^\mu = u^\mu \nabla_\mu \nabla_\beta u_\alpha$$

$$= \left(\nabla_\mu \nabla_\beta u_\alpha + R^i_{\ \alpha\beta\mu} u_i\right) u^\mu$$

$u^\mu \nabla_\mu \nabla_\beta u_\alpha$

$u^\mu \nabla_\mu B_{\alpha\beta}$

$$= - B_{\alpha\mu} B^\mu_\beta - R_{\alpha i \beta\mu} u^i u^\mu$$

$$= - B_{\alpha\mu} B^\mu_\beta - R_{\alpha i\beta\mu} u^i u^\mu$$

(18) <u>Evolution of Expansion.</u>

Take Trace of (17)

$$\frac{D\theta}{d\tau} = \frac{DB}{d\tau} \xrightarrow{\text{scalar}} = \frac{dB}{d\tau} = - B^{\alpha\mu} B_{\mu\alpha} - R_{i\mu} u^i u^\mu$$

(19) $$B^{\alpha\mu} B_{\mu\alpha} = \left(\frac{h^{\alpha\mu}\theta}{3} + \sigma^{\alpha\mu} + \omega^{\alpha\mu}\right)\left(\frac{h_{\mu\alpha}\theta}{3} + \sigma_{\mu\alpha} + \omega_{\mu\alpha}\right)$$

$$= \frac{\theta^2}{3} + \sigma^{\alpha\mu}\sigma_{\mu\alpha} + \omega^{\alpha\mu}\omega_{\mu\alpha}$$

$$= \frac{\theta^2}{3} + \sigma^{\alpha\mu}\sigma_{\alpha\mu} - \omega^{\alpha\mu}\omega_{\alpha\mu}$$

as $h^{\alpha\mu}\sigma_{\mu\alpha} = 0$

$h^{\alpha\mu}\omega_{\mu\alpha} = 0$

$$\therefore \boxed{\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{i\mu}U^iU^\mu}$$

Raychaudhuri Equation,

(20) θ = Fraction change ~~of~~ for the Rate of Cross sectional volume,

→ Rate of Expansion as $|\rightarrow\leftarrow|$ $\theta = 0$

$\frac{d\theta}{d\tau}$ = 2^{nd} Rate of Expn

(21) ~~Focusing Theorem~~

Norm in SR can be $-ve/+ve$.

$$A^iA_i = A^0A_0 + A^1A_1 + A^2A_2 + A^3A_3$$
$$= A^{0^2} - A^{1^2} - A^{2^2} - A^{3^2} = \text{can be } +ve/-ve.$$

(22) But our $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$, θ are Spatial

$\therefore$

$\sigma_{\alpha\beta}\sigma^{\alpha\beta} \overset{*}{=} -ve$

$\omega_{\alpha\beta}\omega^{\alpha\beta} \overset{*}{=} -ve$

As They are Tensorial ∴ in GR they can -ve.

$\theta^2 = +ve$ as $\theta \in R$

(23) Ray chaudri Eqⁿ for $\omega_{\alpha\beta}$, $\sigma_{\alpha\beta}$?

(24) Focusing Theorem

→ Let congruence (Geod & Timelike) be H-S Orth

⇒ $\omega_{\alpha\beta} = 0$

&

→ Imposing Strong Energy Condⁿ

$$R_{\alpha\beta} U^\alpha U^\beta \geq 0$$

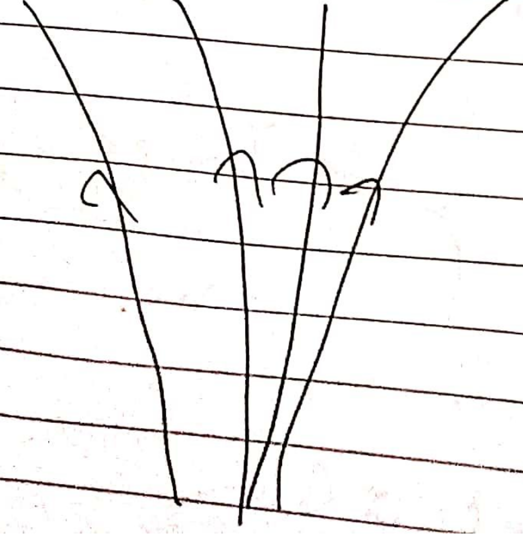
$$\left(T_{\alpha\beta} - \frac{T g_{\alpha\beta}}{2}\right) U^\alpha U^\beta \geq 0$$

Most particles in classical world satisfy Strong Energy Condⁿ

$$\Rightarrow \frac{d\theta}{d\tau} = \frac{\theta^2}{3} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta} U^\alpha U^\beta \leq 0$$

$$\boxed{\frac{d\theta}{d\tau} \leq 0}$$

(25)



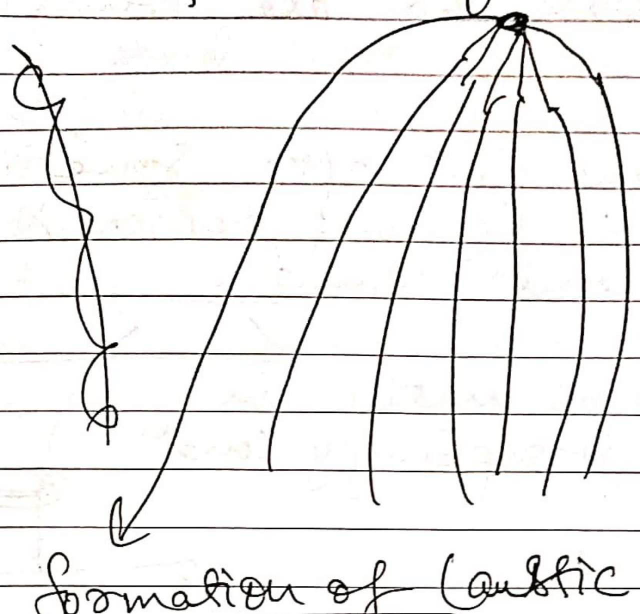
$\theta > 0$ (Exp. Universe)

Rate of Expanding Universe will decrease

∴ Divergence will decrease

(26) Focusing Th. in flat spacetime. $\frac{d\theta}{d\tau} = 0$

(27)



$\theta < 0$
Rate of Cont ↑
as θ already −ve
Convergence ↑

$\boxed{\theta = -\infty} \Rightarrow$ Our Formalism breaks down

(28) Null Geod / Event Horizon

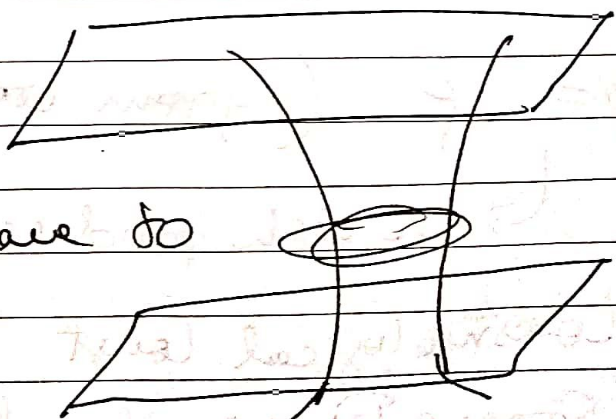
Strong Condition will not work in Null case

Event Horizon is the Null H.S.

Def: Null H.S = $\phi = \text{const}$; $\partial_\alpha \phi \, \partial^\alpha \phi = 0$

$n_\alpha \propto \partial_\alpha \phi$ $\quad n_\alpha n^\alpha = 0$

(29) Wormhole



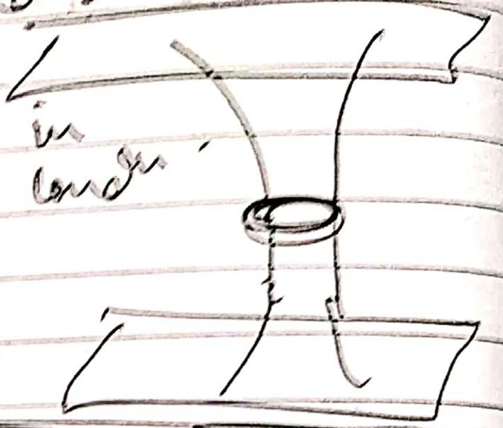
To produce this we have to violate Strong Energy Cond^n.

we can also show that,
to Observers follows H.S. orth. lagrange
$\therefore \omega_{\alpha\beta} = 0$

As
let suppose Observers are coming inside
wormhole

(30) &
~~the~~ Converging But after sometime they
are Diverging $\therefore$ Focusing theorem ✗ $\therefore$
Strong Energy condn violated

$\therefore$ There should be matter in
which violates Strong Energy condn.



(30) This comes from Spacetime Diagram itself
with no Calculations.

(31) Cosmological fluid is Expanding & accelerating, coz our
Universe is expanding & ~~Spa~~ accelerating
$\therefore$ Strong Energy condn violated

Strong Energy condn $\rho + \sum_i p_i \geq 0$; $\rho + p_i \geq 0$
$\therefore$ pressure should be as large as ρ
Energy Density

(32) $p = -\rho$ (from Cosmological Const)
↳ which produces violation in Str. Eng C.

$\therefore$ Cosmological Const leads to Violation of
Strong Energy condn

How to do Cosmology without FRW metric & coordinate?

(33) Cosmology

Cosmological fluid is a eg. of congruence.
u^α; velocity of fluid } we have these qty
θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$ } for Congruence.
In addition to above, as we are talking about fluid we have
ρ, p → from these we can form Energy mom. Tensor for Cosm. fluid

mass Density

$$T^{ab} = \underbrace{\rho u^a u^b}_{\text{Mass Density in Long. Direction along timelike}} + p\underbrace{(g^{ab} - u^a u^b)}_{h^{ab}}$$

pressure in Transverse Direction

Eq of State

(34) Postulating relationship B/w p & ρ i.e. postulat Equation of state.

Postulate: $p = \underset{\text{const.}}{w}\rho$ $\quad p \propto \rho$

If $w = 0$: pressure is neglible compared with Density.
$\Downarrow$
$p = 0$ → Case for Matter Dominated Universe

If $w = \frac{1}{3}$: Radiation field
Case for Radiation Dom. Univ.

If $w = -1$: Dark Energy ≡ Cosmological const.

If $w = -1$ then Universe is accelerating
If $w < -1$ acc. ↑↑ & Singularity at finite time

$B_2 = 0$

(35) $\nabla_\beta T^{\alpha\beta} = 0 \Rightarrow -\frac{d\rho}{d\tau} + (p+\rho)\underbrace{\nabla_\alpha u^\alpha}_{} = 0$ } Along u

$\Rightarrow$ gives eqⁿ which tell how mass Density will change with time along longruence.

$$(\rho+p)\frac{Du^\alpha}{d\tau} + (g^{\alpha\beta}+U^\alpha U^\beta)\partial_\beta p = 0$$ } Orth to U

pressure gradient.

If we assume Cosmolog. fluid follows geodesic

$\therefore \frac{Du^\alpha}{d\tau} = 0$

$\Rightarrow \partial_\beta p = 0$ $\therefore$ pressure is const / uniform inside the fluid.

$\therefore$ No pressure Grad. force

$\therefore$ acc. $= 0$

(36) $-\frac{d\rho}{d\tau} + (p+\rho)\theta = 0$

$$\left(\frac{d\rho}{d\tau}\right) + (\rho\theta)(1+w) = 0$$ } Cosmological Equation

(37) Assumptions:
1. Geodisic Motion
2. Equation of state $p = w\rho$
3. $\sigma_{\alpha\beta} = 0$ No Shear
4. $\omega_{\alpha\beta} = 0$ No Rotation
5. $R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{T g_{\alpha\beta}}{2}\right)$

EFE

$$R_{\alpha\beta}u^\alpha U^\beta = 4\pi(\rho+3p) = 4\pi(1+3w)\rho$$

$\Rightarrow \frac{d\theta}{d\tau} = -\theta^2 - 4\pi(1+3w)\rho ...$

2 Eqn 2 unknowns ρ, a. w is fixed.

(38) This is the way of Doing Cosmology without writing metric

↳ Represent flow in phase space.

(39) Scale factor

$$\theta = \frac{1}{\delta V} \frac{d}{d\tau} (\delta V)$$

Def: $(\delta V) \propto a^3(t)$ (Cross section is prop. to cube of scale factor)

Definition: Scale factor

$$\theta \equiv \frac{1}{a^3} \frac{d}{d\tau} a^3$$

$$= 3 \frac{\dot{a}}{a} = 3H$$

↑ Hubble constant

Putting $\theta \equiv \frac{1}{a^3} \frac{d}{d\tau} a^3$ in our Cosmological eqn

we get to our usual friedman eqn in terms of a.

~~If $w < -\frac{1}{3}$ then Universe is Exp.~~

~~How?~~

$w = -1$ then 2nd term Dom & Univ is Exp.

Date 23 Aug

L-7

① Example: Diverging congruence in flat spacetime in spherical coordinates

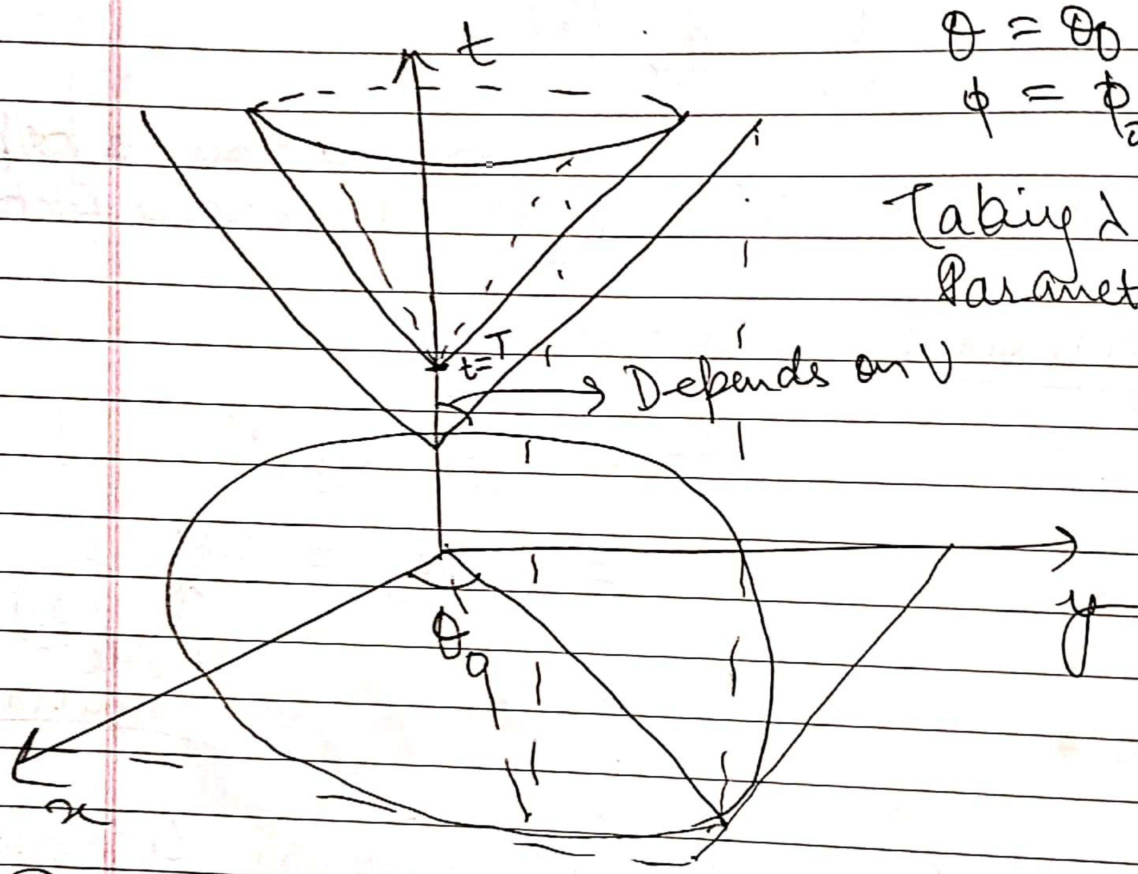
$$ds^2 = c\,dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

Family of timelike Geodesic: $t = T + \lambda$

$r = v\lambda \quad v = \text{Const}$

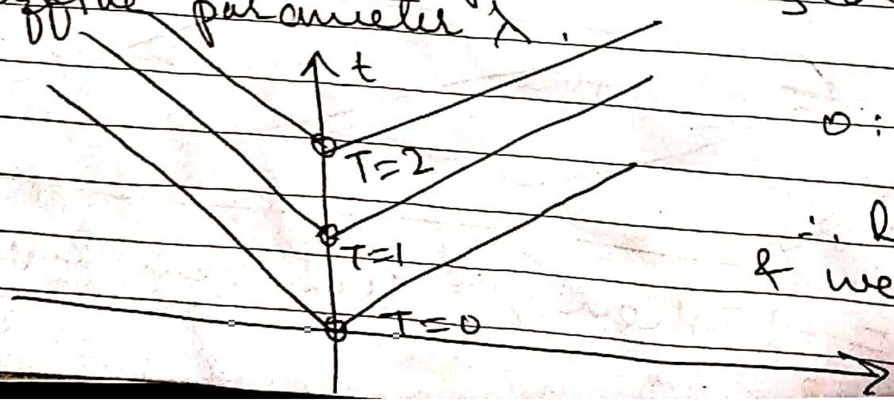
$\theta = \theta_0$ } can
$\phi = \phi_0$ } vary

Taking λ as affine Parameter



② We have 3 parameters (T, ϕ_0, θ_0) to label each Geodesic (congruence) &

each geodesic is parameterized by affine parameter λ.

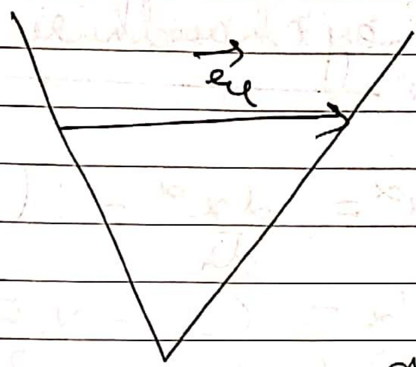


o: Singular points

∴ Remove them & we get congruence

(3) In flat Spacetime

Deviation vector



$$\xi^\beta \nabla_\beta u^a = u^\beta \nabla_\beta \xi^\alpha$$

$$\xi^\beta \partial_\beta u^a = u^\beta \partial_\beta \xi^a$$

Relative vel. is const & zero if they are ||

In Comoving frame $u^\beta = (1, 0, 0, 0)$

$$\xi^\beta \partial_\beta u^a = \boxed{\dot{\xi}^a \neq 0}$$

→ if they are Diverging
if || then $\dot{\xi}^\alpha = 0$

$$\Rightarrow \frac{D^2 \xi^a}{dt^2} = \frac{D}{dt}\left(u^\beta \nabla_\beta \xi^a\right) = u^\gamma \nabla_\gamma \left(u^\beta \nabla_\beta \xi^a\right)$$

$$= u^\gamma \partial_\gamma \left(u^\beta \partial_\beta \xi^a\right)$$

in the comoving frame

$$0 = u^b u^c R^a{}_{bcd}\, \xi^d = \frac{D^2 \xi^a}{dt^2} = \ddot{\xi}^a$$

$$\therefore \boxed{\ddot{\xi}^a = 0}$$

(4) $$\Theta = \frac{1}{V}\frac{dV}{d\lambda} = \frac{1}{\pi \xi^3}\frac{d\, \pi \xi^3}{d\lambda}$$

$$= \frac{1}{\xi^3}\frac{d \xi^3}{d\lambda} = 3 \frac{\dot{\xi}}{\xi}$$

const
incr

$$\frac{d\Theta}{d\lambda} = \cancel{3\frac{\ddot{\xi}}{\xi}} - 3\left(\frac{\dot{\xi}}{\xi}\right)^2 = -ve.$$

$\therefore$ Focusing theorem is satisfied

(5) Raychaudhuri eqn in flat Sp. Time

$$u^\alpha = \frac{dx^\alpha}{d\lambda} = (1, v, 0, 0)$$

$$u_\alpha = (1, -v, 0, 0) = \eta_{\alpha\beta} u^\beta$$

$$u^\alpha u_\alpha = 1 - v^2 = \text{Not Normalized}$$

(6) We can Normalize this By taking proper time τ as parameter

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma \frac{dx^\alpha}{dt} = \gamma(1, v, 0, 0)$$

$$u^\alpha u_\alpha = \gamma^2 - \gamma^2 v^2 = 1$$

see (1)

(7) But Rn we take λ as affine parameter to shift to Null case Easily.

(8) As $u_\alpha = (1, -v, 0, 0)$

normal to Hypersurface

$\phi = t - vx =$

$$u_\alpha = \partial_\alpha (t - vx) = \partial_\alpha f$$

$$= \text{Gradient of } f^n$$

$\therefore u_\alpha \propto \partial_\alpha f \Rightarrow$ Congruence is Hyp. Orthog

$$\therefore \omega_{\alpha\beta} = 0.$$

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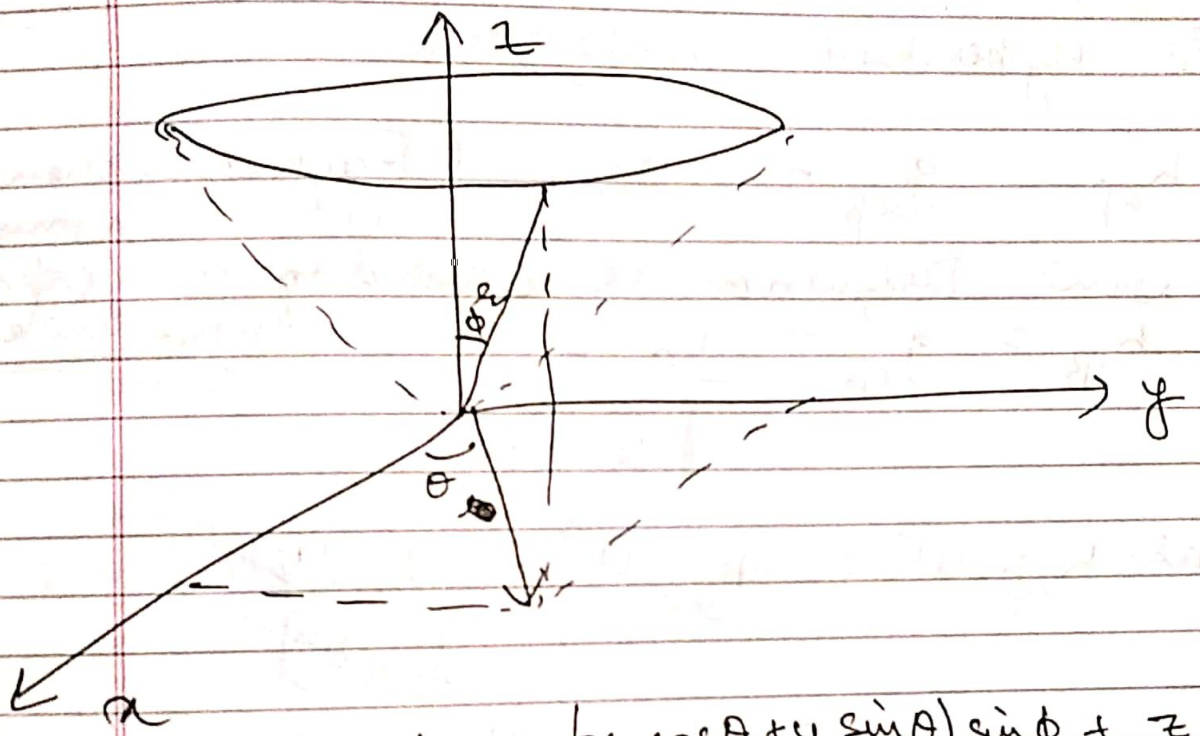
(9) $\nabla_\beta U_\alpha = B_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & vr\sin^2\phi & 0 \\ 0 & 0 & 0 & vr \end{bmatrix}$



$$r = (x\cos\theta + y\sin\theta)\sin\phi + z\cos\phi$$

wrt $B_{\alpha\beta} = \partial_\beta \partial_\alpha \left(t - v\left((x\cos\theta + y\sin\theta)\sin\phi + z\cos\phi\right)\right)$

$$= \partial_\beta \partial_\theta \left(-v\left((x\cos\theta + y\sin\theta)\sin\phi\right)\right)$$

$$= \partial_\beta \left(-v\left((-x\sin\theta + y\cos\theta)\sin\phi\right)\right)$$

$$= \partial_\theta \left(-v\left((-x\sin\theta + y\cos\theta)\sin\phi\right)\right)$$

$$B_{\theta\theta} = v\,(r\sin\phi)\sin\phi = vr\sin^2\phi$$

$$B_{\phi\phi} = vr$$

(10) $B_{\alpha\beta}$ is spherical symmetric as ϕ isn't the
$B_{\alpha\beta}$ is sym as only Diag.

$B_{\alpha\beta} = B_{\beta\alpha} \Rightarrow \omega_{\alpha\beta} = 0$

$\therefore$ Hypersurf. Orthogonal.

(11) $h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$ (Earlier when U^α was Norm
Now Definition is adjusted to make prop. of $h_{\alpha\beta}$ similar

$$h_{\alpha\beta} = g_{\alpha\beta} - \frac{U_\alpha U_\beta}{(U_\alpha U^\alpha)}$$

as $h_{\alpha\beta} U^\beta = g_{\alpha\beta} U^\beta - \frac{U_\alpha \cancel{(U_\beta U^\beta)}}{\cancel{(U_\alpha U^\alpha)}}$

$= U_\alpha - U_\alpha = 0$

(12) $B_{\alpha\beta} = \frac{h_{\alpha\beta}}{3}\theta + \sigma_{\alpha\beta}$

$\theta = h^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = \nabla_\beta U^\beta = \frac{2v}{r}$

as $B_{\alpha\beta}$ is Orth to U^α; $B_{\alpha\beta} U^\alpha = 0$

as $B_{\alpha\beta} U^\alpha U^\beta \stackrel{*}{=} B_{00} \stackrel{*}{=} 0$ $\rightarrow$ in comoving frame

$\therefore B_{\alpha\beta} U^\alpha U^\beta \stackrel{*}{=} 0$

$$\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{2}{3}\frac{v^2}{r^2}$$

see (12).

as θ depends only on r

$$\frac{d\theta}{d\lambda} = u^\alpha \partial_\alpha \theta = \partial_t \theta + v\partial_r \theta$$

$$= 0 - \frac{2v^2}{r^2} = -\frac{2v^2}{r^2}$$

By Ray. eq^n

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{3} - \sigma_{\alpha\beta}\sigma^{\alpha\beta}$$

$$= -\frac{2v^2}{r^2}$$

Verified

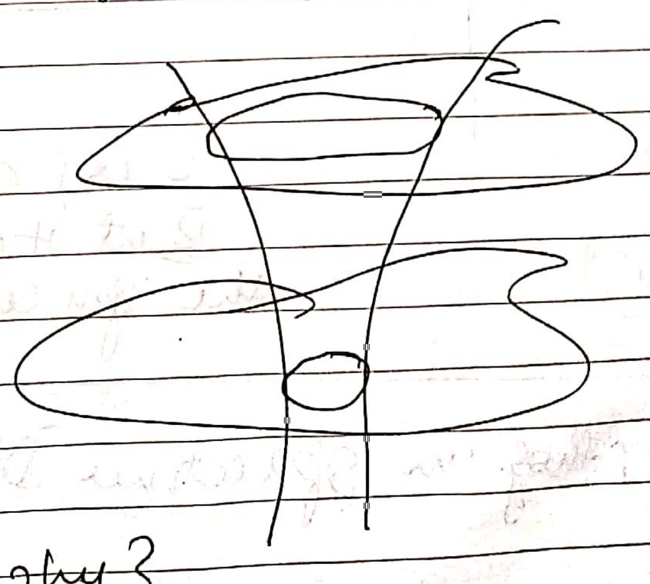
see (4)

(13) In SR we can have congruence where $\omega_{\alpha\beta} \neq 0$; No hyp. orth.

But still $\frac{d\theta}{d\lambda} < 0$

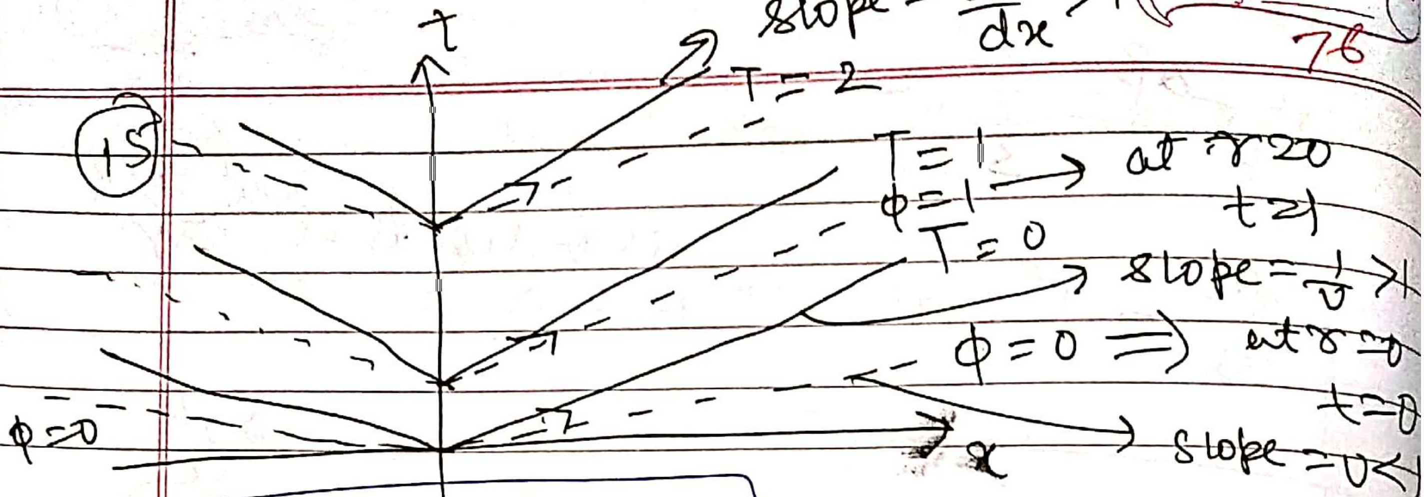
(14) Why shear term comes here?

Shear



Here why?

15



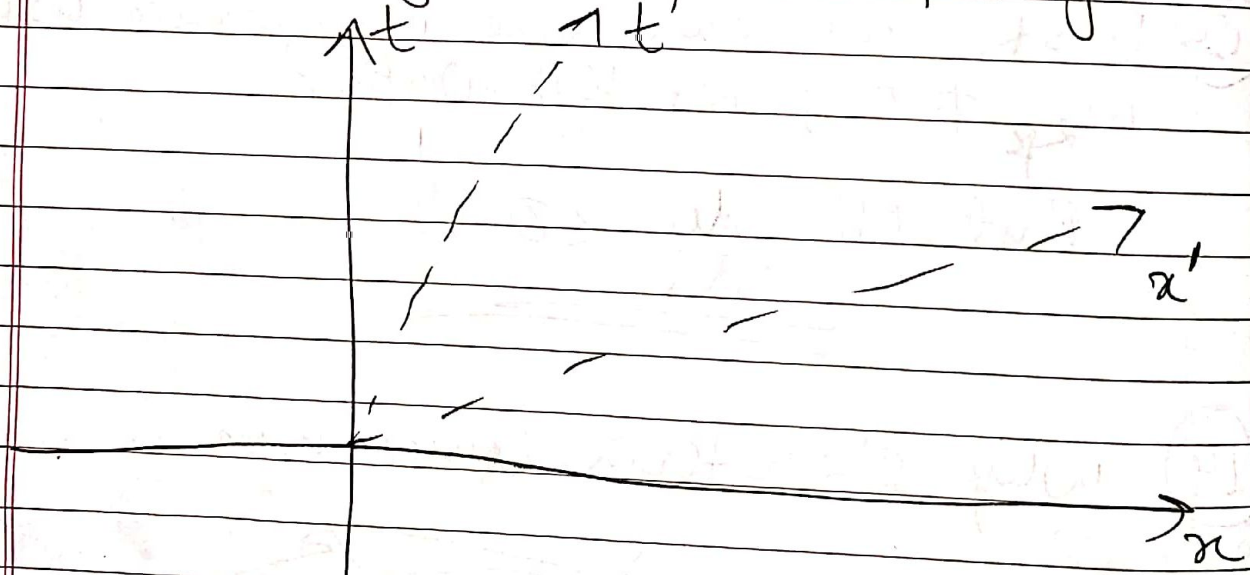
H.S. is spacelike always?

$\phi = t - vr$

Hyper surf. $\phi = t - vr = \text{const}.$

But as $u_\alpha = \partial_\alpha \phi$ ∴ Cong is Hyp orth.

It doesn't look in the Diag. that they are orthog. But in reality they are.



t' is Orth. to x'
But it doesn't look in the spacetime Diag.

Orthog in spacetime Diag?

Why Curvature ≠ 0

(16) Metric on the hyper surf : Induced Metric

On each Hyp. $t = vr + \text{const}$

line element $dt = v\,dr$

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$ds^2\Big|_{H.S.} = v^2 dr^2 - dr^2 - r^2 d\Omega^2$$

$$ds^2\Big|_{\text{Hypsurf.}} = \underbrace{(v^2-1)\,dr^2 - r^2 d\Omega^2}$$ → 3D metric on H-S.

$dr^2 - r^2 d\Omega^2$ Represents flat But here $(v^2-1)dr^2$ can't be converted to flat

∴ This Represents Curved spacetime

if $r^2 d\Omega^2$ is there then flat spacetime.

(17) As this Represents Curved ∴ $R \neq 0$

$$R^{\theta\phi}{}_{\theta\phi} = \frac{-v^2}{(1-v^2)\,r^2}$$

∴ $ds^2\Big|_{\text{Hyp.}}$ is 3D surf. with Intrinsic curvature

at $r=0$ Singularity as $R^{\theta\phi}{}_{\theta\phi} = \infty$

But we have exclude $r=0$ line

Why Curved H.S. ?

(18) Taking Null Case

$\text{slope} = \frac{1}{v} \longrightarrow 1$ [Timelike → Null like

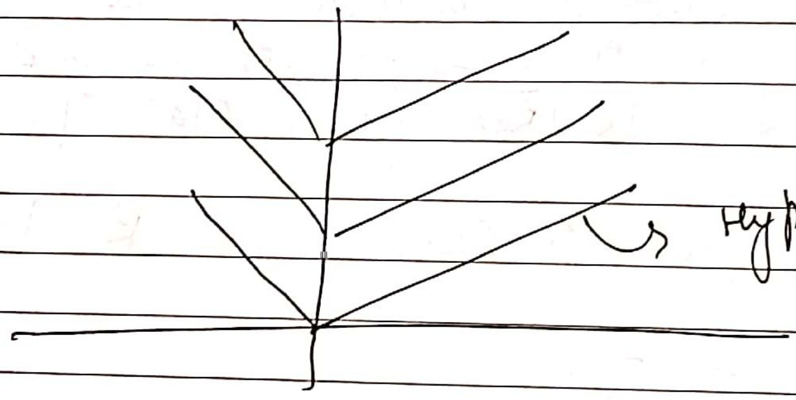
$\text{Slope} = v \longrightarrow 1$ (Hypersurf)

In the limit they coincide.

$u_\alpha = (1, -v \to 1, 0, 0)$

But as $t = vr + \text{const}$; $u_\alpha = \partial_\alpha (t - vr)$
& they are orth. to congruence.

∴ In the limit too they will be Orth



(19) Null geod are orth to Hyp.
& Null geod are Tangent to Hyp.

Orth : $A^\alpha B_\alpha = 0 \Rightarrow \vec{A} \perp \vec{B}$

for Null $k^\alpha k_\alpha = 0 \Rightarrow \vec{k} \perp \vec{k}$
as $k_\alpha = \partial_\alpha (t - vr)$

∴ $k \parallel \Sigma$ ~~∴ $k \perp \Sigma$~~
But $k \perp k$ ~~But $k \perp k$~~
∴ $k \perp \Sigma$ ~~∴ $k \parallel \Sigma$~~ } ∴ $k \parallel \Sigma$
~~$k \perp \Sigma$~~

$A^{\alpha}B_{\alpha} = 0$

$K^{\alpha} = (1, 1, 0, 0)$

$K^{\alpha} = (1, 0, 0, 0)$

(20) $u^{\alpha} = (1, v) = \frac{dx^{\alpha}}{d\lambda}$

$\therefore U_{\alpha}U^{\alpha} = 1 - v^2$

If Normalization is done.

$u^{\alpha} = \frac{dx^{\alpha}}{d\tau} = \gamma^2 (1, v \to 1) = \gamma^2 (1, 1)$

$u^{\alpha} = (\gamma^2, \gamma^2)$ where $\gamma \to \infty$

Singularity Tangent vector Not Defined

(21) But in our case.

$u^{\alpha} = (1, v) = \frac{dx^{\alpha}}{d\lambda} \to$ affine

$u^{\alpha}u_{\alpha} = 0$

$u_{\alpha} = \partial_{\alpha}\phi$

(22) when $v \to 1$

$ds^2\Big|_{Hyp} = \underbrace{(v^2 - 1)}_{\downarrow 0} dr^2 - r^2 d\Omega^2$

spatial

$\therefore$ Loosing one Dimension

$ds^2 \longrightarrow -r^2 d\Omega^2$ (metric Becomes degenerate)

0 Eigenvalues (metric becomes 2 Dim)

Earlier we had $\neq 0$ Eig. values

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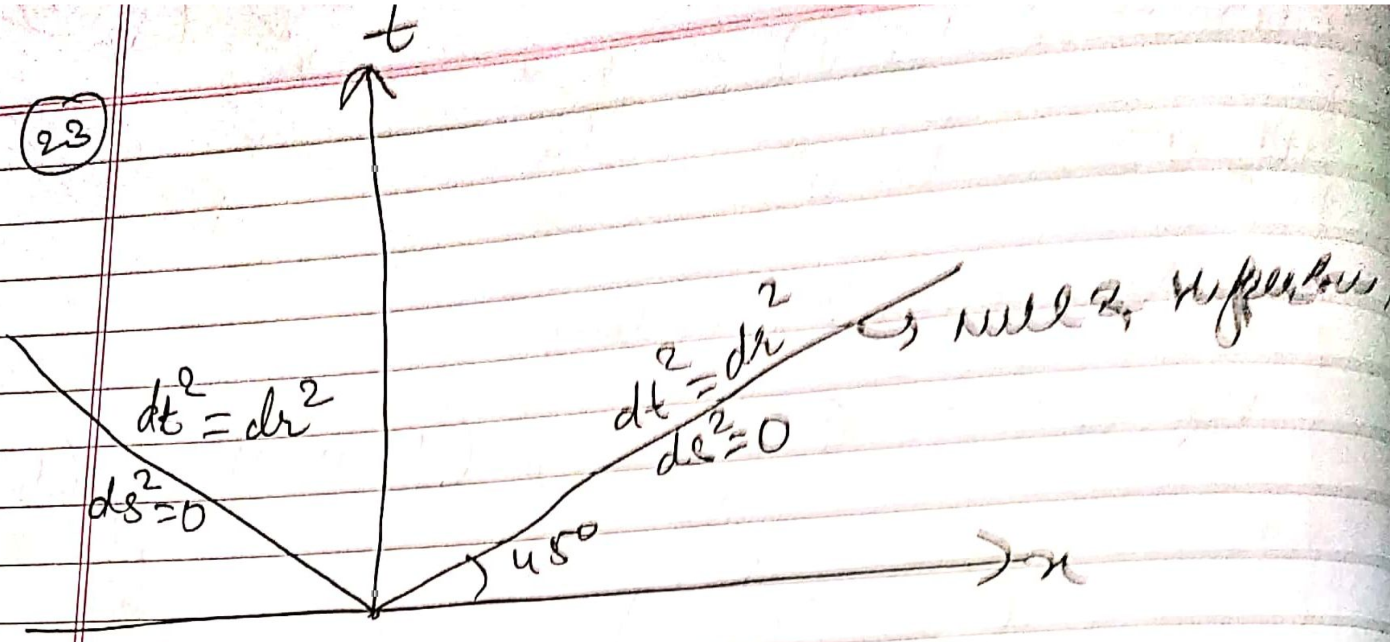


from (16) $\quad dt = v\,dr \qquad v \to 1$

$$dt = dr$$

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$$

$$ds^2\big|_{Hyp} = \cancel{-r^2 d\Omega^2}\; (v^2-1)\,dr^2 - r^2 d\Omega^2 \Rightarrow 3D$$

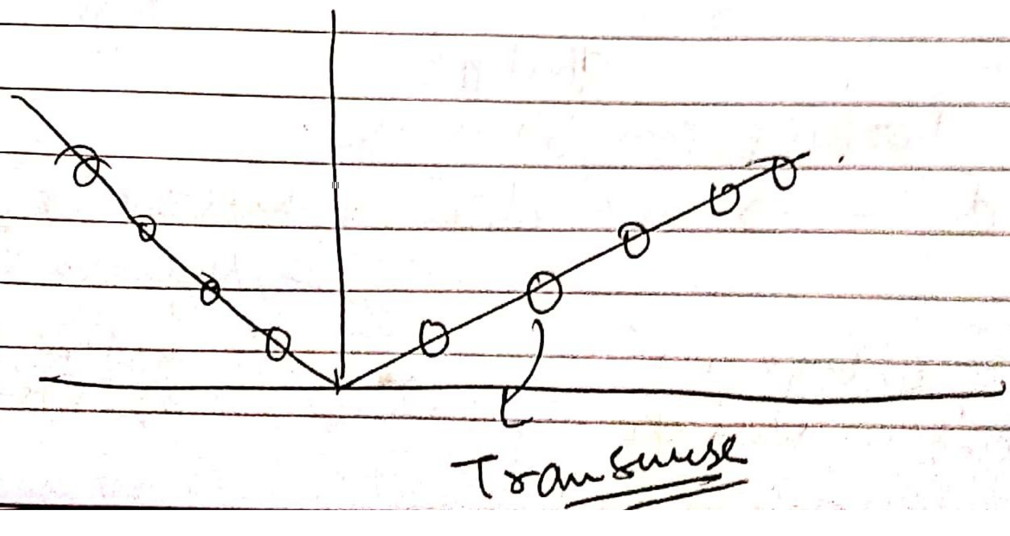
$v \to 1$

$$ds^2\big|_{Hyp} = -r^2 d\Omega^2$$

∴ Two Null direction along which

$$ds^2 = 0$$

∴ 2D Transverse space where $ds^2 \neq 0$



How to draw 2D Transverse Space?

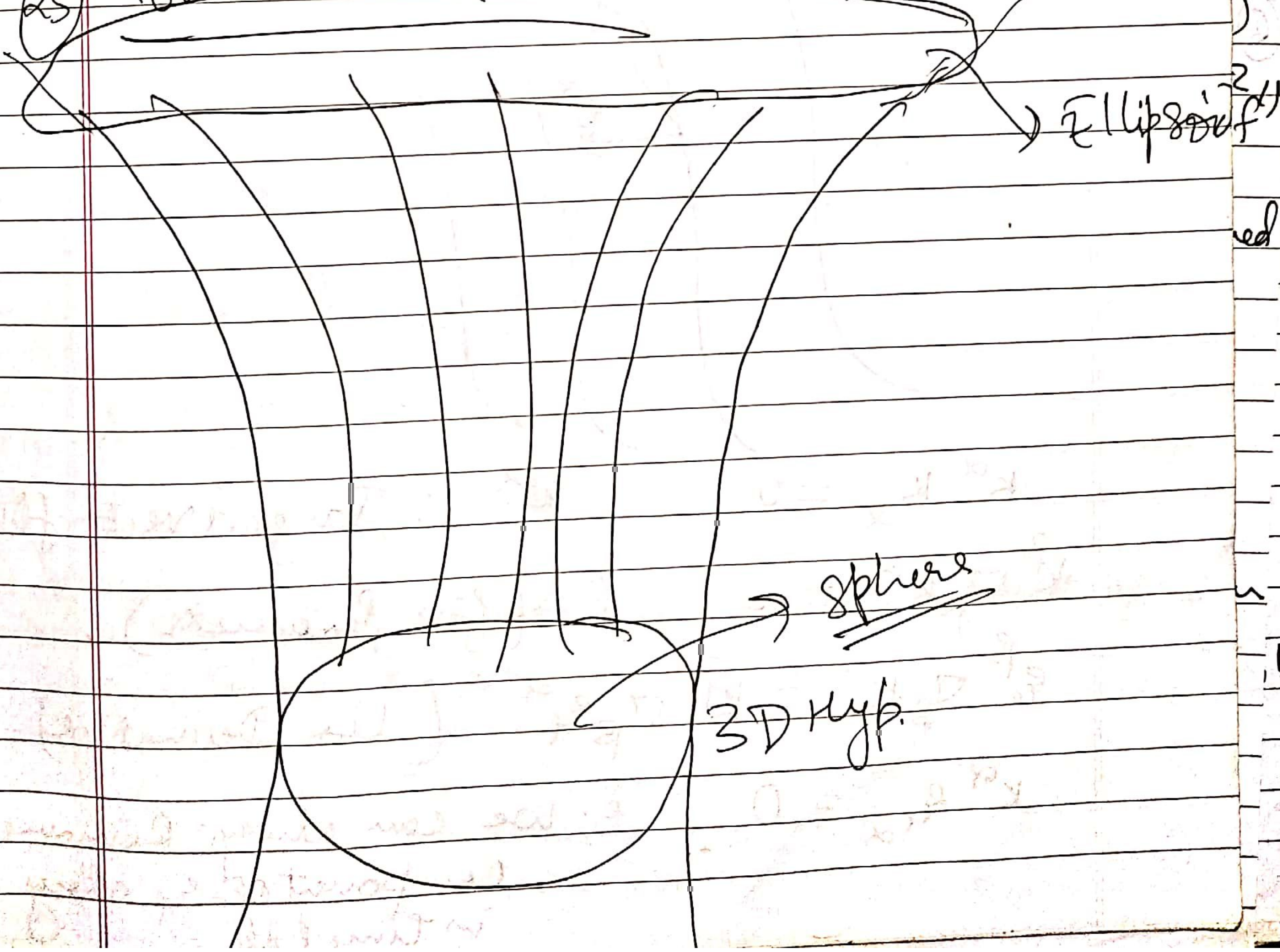
(24) In the Timelike Geodesic

Hypersurface is 3D & $\theta = \frac{1}{V}\frac{dV}{d\lambda}$

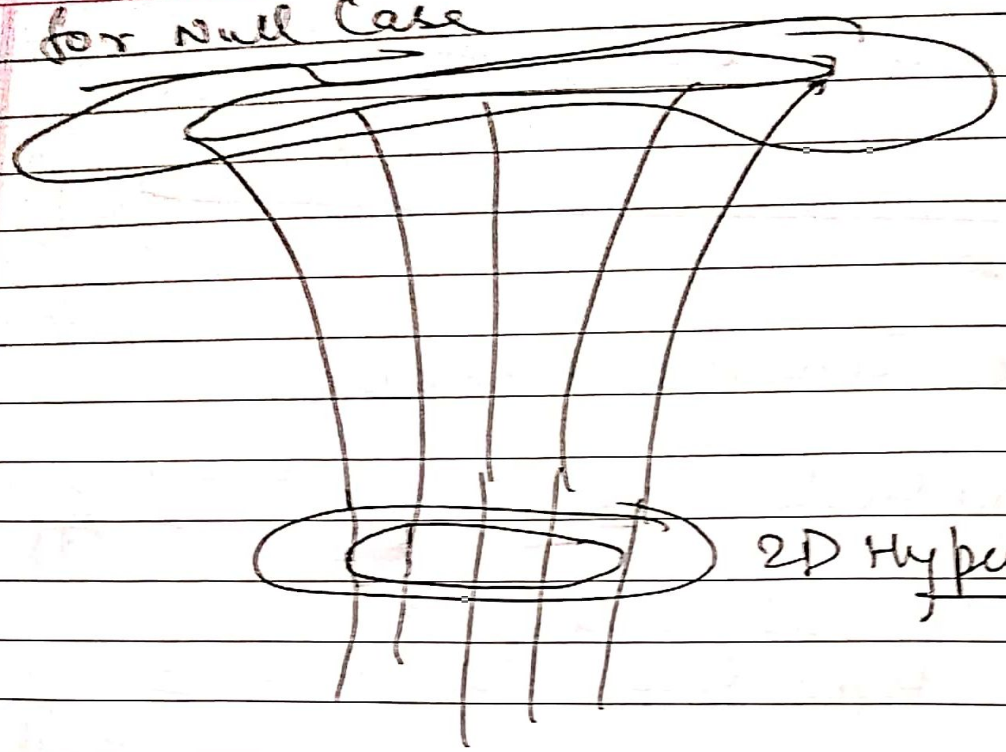
But in Null case

Hypersurface is 2D, & $\theta = \frac{1}{A}\frac{dA}{d\lambda}$.

(25) For Timelike Geodesic

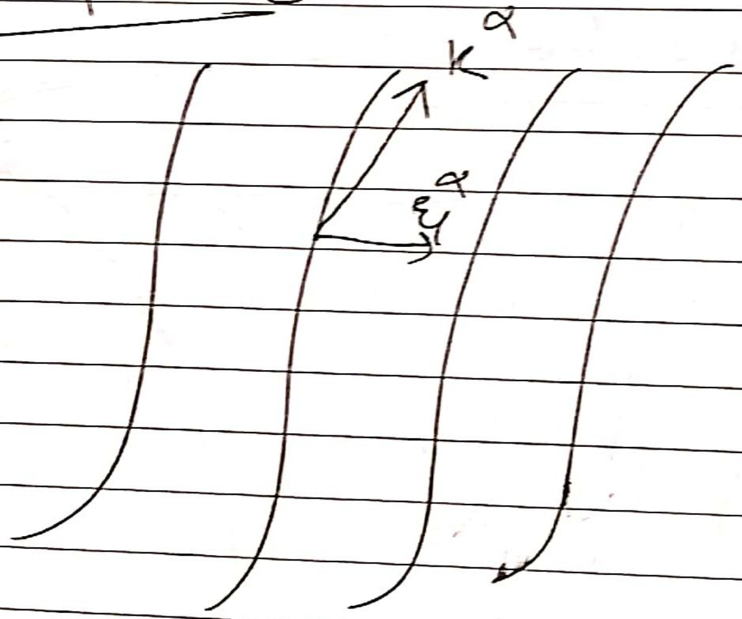


for Null Case



2D Hyper Surface

(2) Null Geodesic



$k^\alpha k_\alpha = 0$ $\quad$ k^α : Tangent vector field

$k^\beta \nabla_\beta k^\alpha = 0$ $\quad$ (Affine Parameter)

$\xi^\beta \nabla_\beta k^\alpha = k^\beta \nabla_\beta \xi^\alpha$ $\quad$ (Lie Derivative)

$k^\alpha \xi_\alpha = 0$ $\quad$ i.e. we can always Remove component of ξ along k

Imp

$$\boxed{K^\alpha B_{\alpha\beta} = B_{\alpha\beta} K^\beta = 0}$$ (Same Calculation as in Timelike Geod)

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Nullike:

$\Rightarrow K^\alpha B_{\alpha\beta} = K^\alpha \nabla_\beta K_\alpha = \frac{1}{2}\nabla_\beta(K^\alpha K_\alpha) = 0$

$\Rightarrow K^\beta \nabla_\beta K_\alpha = 0$ (Geod).

Now ~~if $e_\alpha = (ak_\alpha, 0, 0, 0)$~~

~~then $k^\alpha e_\alpha = 0 \Rightarrow k$~~

if $e_\alpha = a K_\alpha$

then $K^\alpha e_\alpha = a K^\alpha K_\alpha = 0$

$\therefore$ If $k^\alpha e_\alpha = 0$ doesn't kill of component of e along k But it doesn't.

Transverse metric

(27) Timelike Geod

$h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$ → Normalized

& If taken to Null case It will not work due to di

& $h_{\alpha\beta} U^\beta = U^\alpha h_{\alpha\beta} = 0$

What about null case?

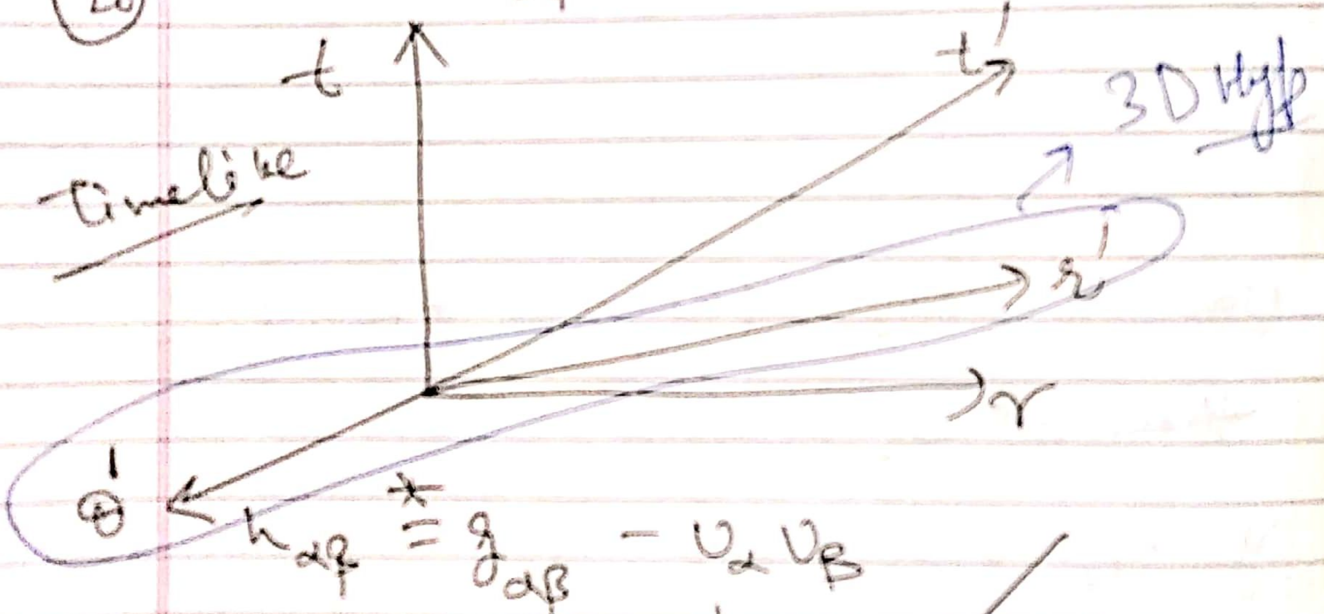
$h_{\alpha\beta} = g_{\alpha\beta} - K_\alpha K_\beta$ X as K_α is Not Defined

$h_{\alpha\beta} K^\beta \neq 0 \neq K_\alpha$

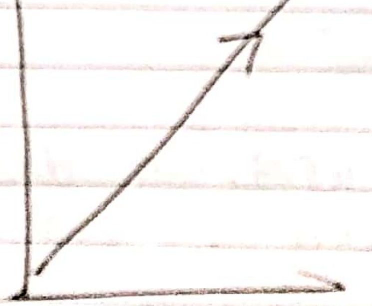
→ $h_{\alpha\beta}$ has to be 2D

(28)

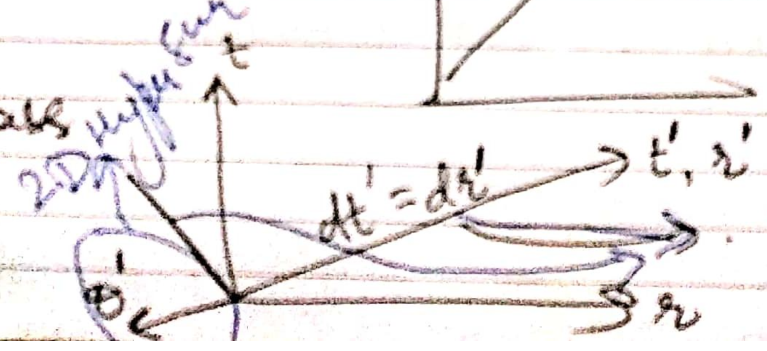
Timelike



$h_{\alpha\beta} \stackrel{*}{=} g_{\alpha\beta} - U_\alpha U_\beta$



in Null case



∴ 2D has to be Rie...

Removing 2 Null Directions.

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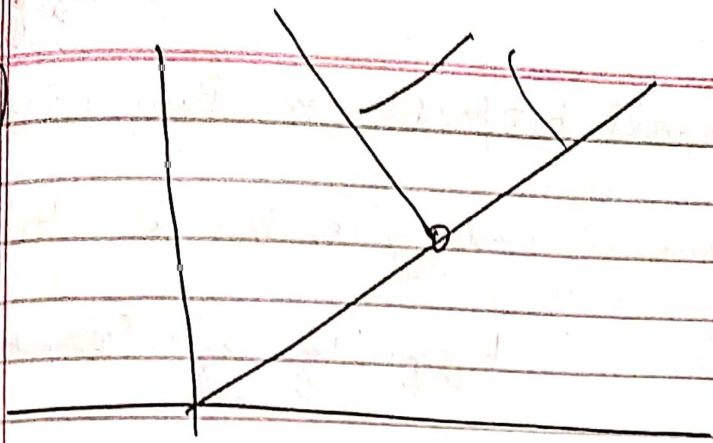
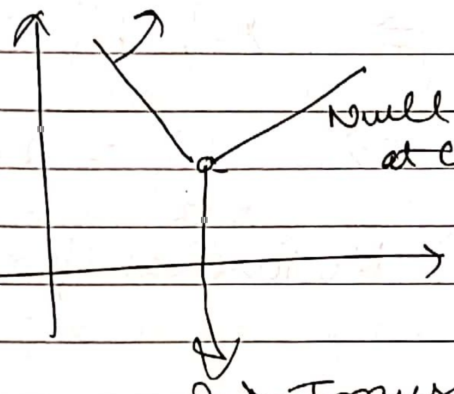
(29)

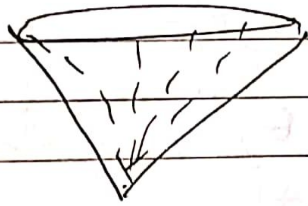


(30) in flat spacetime



We already have one Null Direction given to us we have to obtain another one (Auxilary) & then Remove Both & get 2D Transuer space.

⇒ Auxiulary can be taken any



Given k^α
Pick N^α (Auxilary)
should be Different from k^α

⇒ as N^α is Null vector
$$N^\alpha N_\alpha = 0$$

But why -1 ?

⇒ As N^α is Null vector we can Normalize it arbitrarily
$$N^\alpha k_\alpha = +1 \quad ?$$
Positive → the as they are future Directed
0

Then
$$\boxed{h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta}$$

(21) It $h_{\alpha\beta}$ holds all properties of Projection op.

$\Rightarrow h_{\alpha\beta} k^\beta = 0 \quad \therefore h_{\alpha\beta}$ is Transv. to k^β

$\Rightarrow h_{\alpha\beta} N^\beta = 0 \quad \therefore h_{\alpha\beta}$ is 2D Transv. to k^β, N^β.

$\Rightarrow h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$

$\Rightarrow h^i{}_i = 2$

(22) Motivation for $h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$

$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$

we have to remove both the

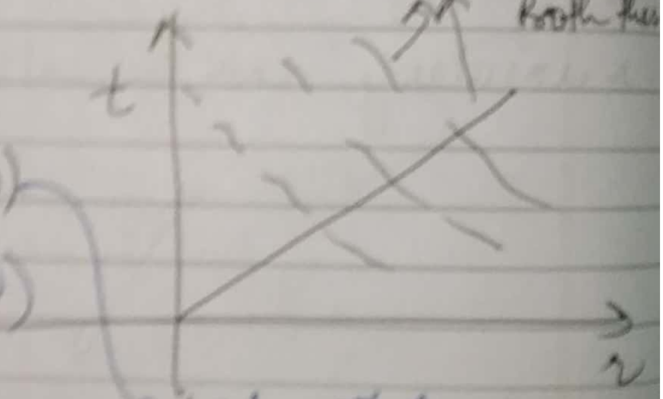
$k^\alpha = (1,1,0,0)$

Pick N^α s.t. $k^\alpha N_\alpha = +1$

Let $N^\alpha = c(1,-1,0,0)$

we can also choose

$N^\alpha = c(1,-1,0,0)$



$k^\alpha N_\alpha = c(-2) = +1$

$\therefore c = \frac{-1}{2}$

Let N^α lie in t-r plane

$\therefore N_\alpha = \frac{-1}{2}(1,-1,0,0)$

$\therefore h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$

$= \text{Diag}(0,0,-r^2,-r^2\sin^2\theta)$

Det = 0

$\therefore h^{\alpha\beta} g_{\alpha\beta}$ Not Defined

$\therefore$ 2D Transverse space.

$B_{\alpha\beta} = h_{\alpha\beta} \rightarrow$ how?

Shear & Rotation = 0

$$\theta = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{d}{d\lambda}(4\pi r^2) = \text{Fractional Rate of change of Cross Sectional Area}$$

(33) ∴ Null vectors brings that Transverse space has to be 2D.
& to do that we have to take another Auxillary Null vector to get metric of Transverse spac.

(34) How to choose Auxillary Vector?

$$\left.\begin{aligned} N^\alpha N_\alpha &= 0 \\ N^\alpha K_\alpha &= 1 \end{aligned}\right\}$$ 2 Conditions to choose vector

∴ Under Constraint.

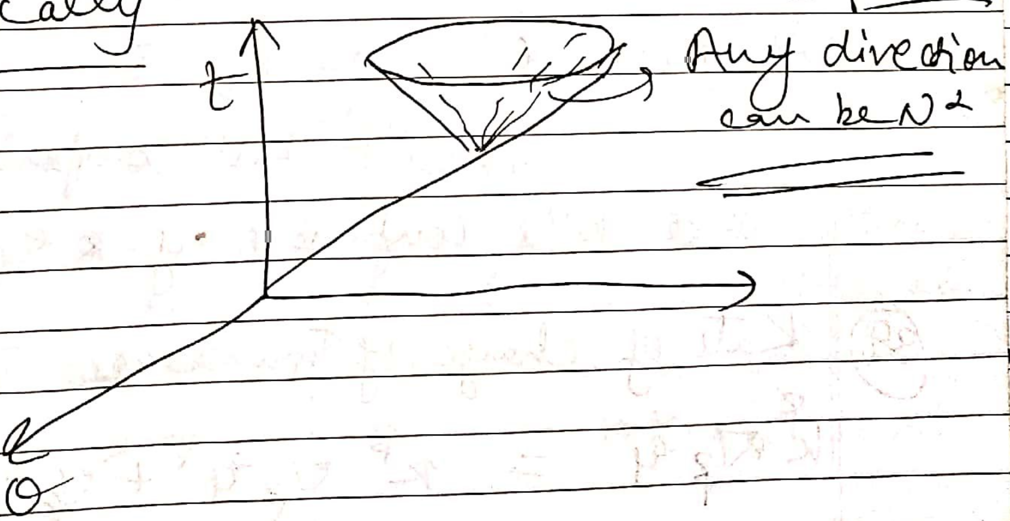
why Under Constt? → as 4 comp & 2 const only

∴ There is not one Unique way of choosing N^α.

There are many ways & each way brings out different $h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$

∴ $h_{\alpha\beta}$ also not Unique. ∃ There are many other thing which are not affected by $h_{\alpha\beta}$ being not Unique eg. θ.
Expansion

(35) Geometrically



(36) $B_{\alpha\beta} = \nabla_\beta K_\alpha$

$K^\beta \nabla_\beta \xi^\alpha_4 = B^\alpha_\beta \xi^\beta_4$

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Earlier in Timelike $\xi^\alpha_\varphi u_\alpha = 0$
ξ^α_φ was already in (...)
But Not Now

(37) Project ξ^α_φ into Transverse Subspace $\tilde{\xi}^\alpha_\varphi$

$$\tilde{\xi}^\alpha_\varphi = h^\alpha_\beta \xi^\beta_\varphi \quad \therefore \text{ Components along } N^\alpha \text{ \& } k^\alpha \text{ removed}$$

$$\tilde{\xi}^\alpha_\varphi = (\delta^\alpha_\beta + k^\alpha N_\beta + N^\alpha k_\beta)\, \xi^\beta_\varphi$$

$$= \xi^\alpha_\varphi + \underbrace{k^\alpha (N_\beta \xi^\beta_\varphi)}_{\text{comp. along } k^\alpha} + \underbrace{N^\alpha (k_\beta \xi^\beta_\varphi)}_{\text{comp along } N^\alpha}$$

has comp along k & N^α
as $\tilde{\xi}^\alpha_\varphi$ along Transverse
$\therefore$ It cancels comp. along k^α & N^α

But $k_\beta \xi^\beta_\varphi = 0$ (in the starting)

$$\therefore \tilde{\xi}^\alpha_\varphi = \xi^\alpha_\varphi + k^\alpha (N_\beta \xi^\beta_\varphi)$$

$\therefore$ ξ^α_φ has Nothing along N^α
It only canceles k^α component

(38) As from (26) $\xi^\alpha_\varphi k_\alpha = 0$

Doesn't kill component of ξ_φ along k
But kills component of ~~k~~ ξ_φ along N.

(39) Rate of change of Transverse Deviation

$$k^\beta \nabla_\beta \tilde{\xi}^\alpha_\varphi = k^\beta \nabla_\beta \xi^\alpha_\varphi + \nabla_\beta (N_i \xi^i_\varphi)\, k^\beta k^\alpha$$
$$+ (N_i \xi^i_\varphi)\, \cancel{k^\beta \nabla_\beta k^\alpha}$$

All the previous thing we did in flat spacetime. WB curved spacetime?

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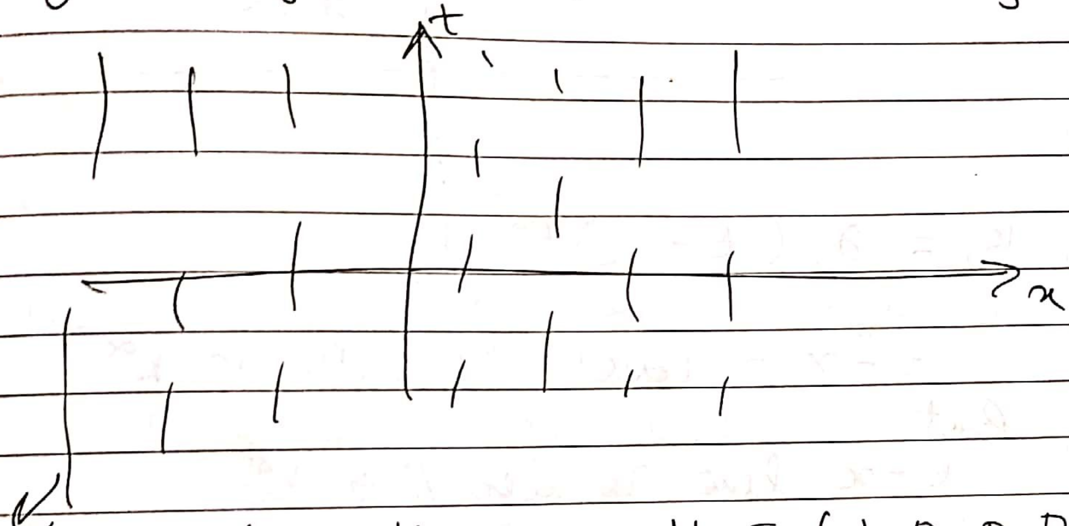
L-8

(1) Flat spacetime

Cartesian coord.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Congruence of Timelike Geod. where $dx = dy = dz = 0$



Hypersurf. Orthog. $u_\alpha = (1, 0, 0, 0)$

$$u_\alpha = \partial_\alpha (t)$$

$f = t = \text{const}$

$\therefore$ Congruence is Hyp. Orth.

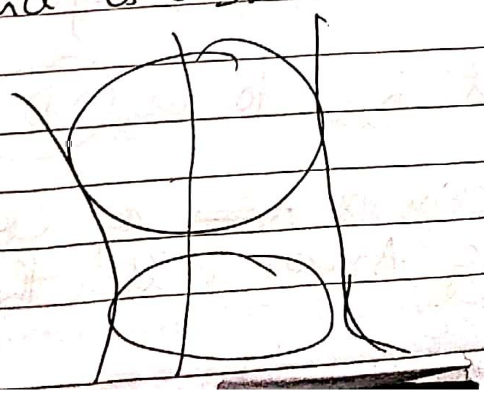
(2) If I need to see ~~metric~~ line element on that hypersurf we need to remove dt^2.

$\therefore$ My induced line element is 3D then

$\therefore$ Hypersurf is 3D & cross section of congruence is 3D then

But flat H.S

$R_{ij} \neq 0$



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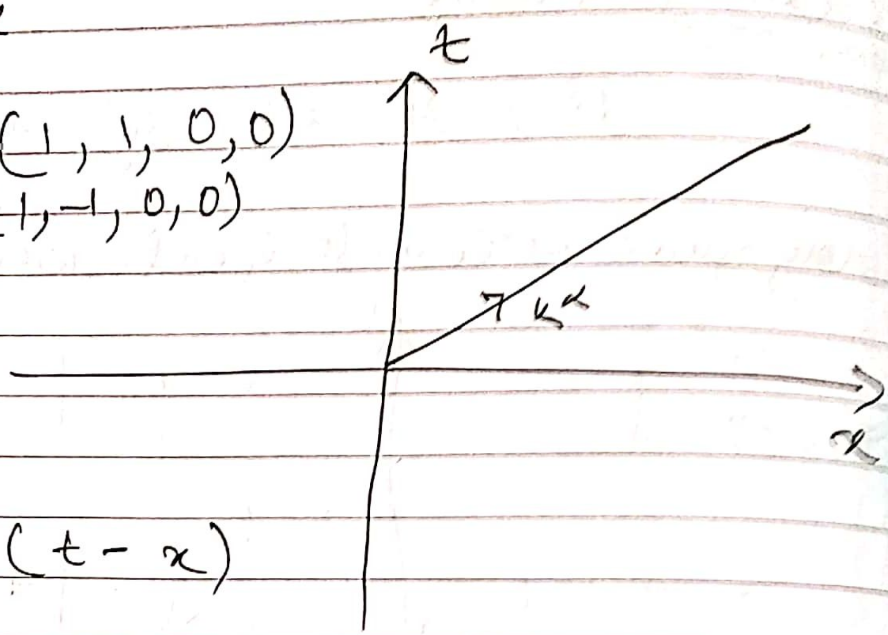
(3) Induced metric $h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$

(4) Null Case

$k^\alpha = (1, 1, 0, 0)$
$k_\alpha = (1, -1, 0, 0)$



$k_\alpha = \partial_\alpha (t - x)$

$t - x =$ const. is orth to k^α

But

$t - x$ line is also // to k^α

$\therefore$ Hyp. is Orth & // to k^α.

(5) Hypersurface is $\phi = t - x$

$dt = dx$

$\therefore \; ds^2\Big|_{Hyp} = -dy^2 - dz^2 \Rightarrow$ (2D)

2D Bec. Degeneracy comes from Null Direction as Null Surfaces ~~have~~ ~~are~~ have degenerate line element

$\therefore$ We expect $h_{\alpha\beta}$ to be (2D).

& Cross sections ~~Area~~ of Congruence would be Areas rather than volume.

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(6) We have to kill 2 null direction from spacetime to reach 2D $h_{\alpha\beta}$ N_α & k_α

$\therefore$ $h_{\alpha\beta} = g_{\alpha\beta} - k_\alpha N_\beta - N_\alpha k_\beta$

(7) N_α is not unique which makes $h_{\alpha\beta}$ is not unique.

(8) $k^\alpha B_{\alpha\beta} = B_{\alpha\beta} k^\beta = 0$

k is orth to $B_{\alpha\beta}$

But $B_{\alpha\beta} N^\beta \neq 0$ & $N^\alpha B_{\alpha\beta} \neq 0$

$\therefore$ $B_{\alpha\beta}$ is not orth to N

(9) $\tilde{\xi}^\alpha = h^\alpha_i \xi^i$ = Transverse part of ξ^i = Purely Transverse Deviation vector

$k^\beta \nabla_\beta \tilde{\xi}^\alpha = \nabla_\beta (h^\alpha_i \xi^i) k^\beta$

$= (\nabla_\beta h^\alpha_i) \xi^i k^\beta + (\nabla_\beta \xi^i) h^\alpha_i k^\beta$

$= \nabla_\beta (\delta^\alpha_i - k^\alpha N_i - N^\alpha k_i) \xi^i k^\beta + (\nabla_\beta \xi^i) h^\alpha_i k^\beta$

How do I know this has comp. in k direction?

$= -k^\alpha (k^\beta \xi^i \nabla_\beta N_i) - \cancel{(k_i \xi^i)} k^\beta \nabla_\beta N^\alpha + (\nabla_\beta \xi^i) h^\alpha_i k^\beta$

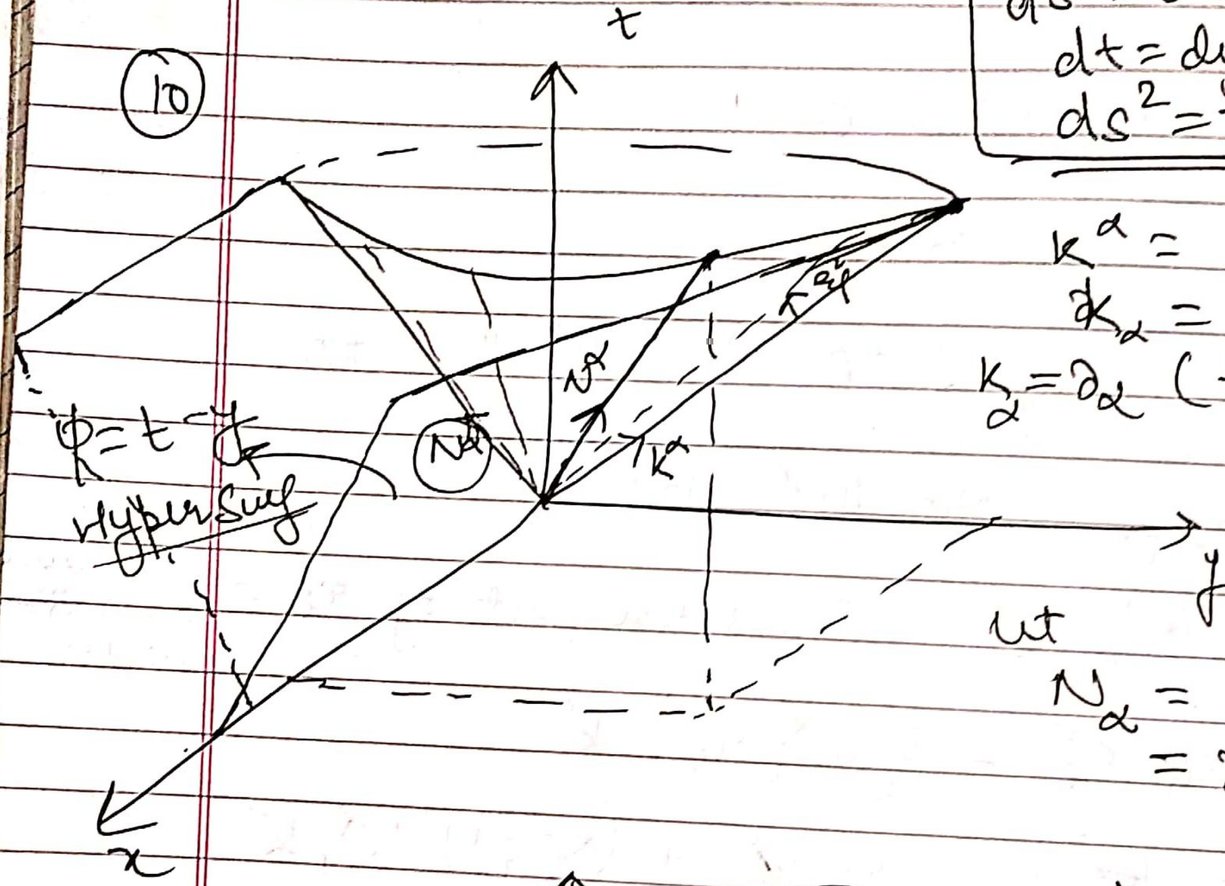
Component along k

But $k^\beta \nabla_\beta \xi^i = \xi^\beta \nabla_\beta k^i$

$\therefore$ $\underbrace{k^\beta \nabla_\beta \tilde{\xi}^\alpha}_{\text{Transverse velocity 2 neighbouring geodesic.}} = \xi^\beta h^\alpha_i (\nabla_\beta k^i) - k^\alpha (k^\beta \xi^i \nabla_\beta N_i)$

Light in y direction only

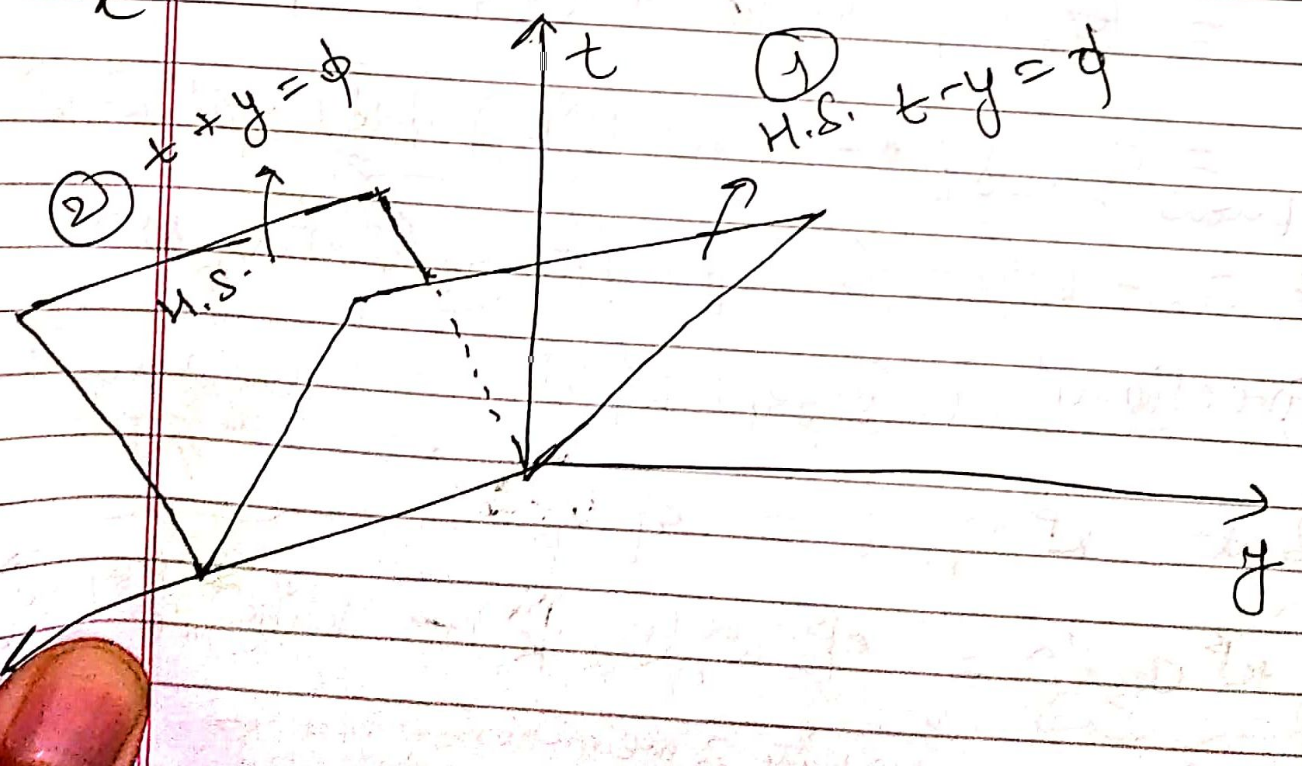
$$ds^2 = dt^2 - dx^2 - dy^2$$
$$dt = dy \quad \text{on Hyper}$$
$$ds^2 = -dx^2$$



$$k^\alpha = (1, 0, 1)$$
$$k_\alpha = (1, 0, -1)$$
$$k_\alpha = \partial_\alpha (t - y)$$

Let

$$N_\alpha = (1, 0, 1) = \partial_\alpha (t + y)$$

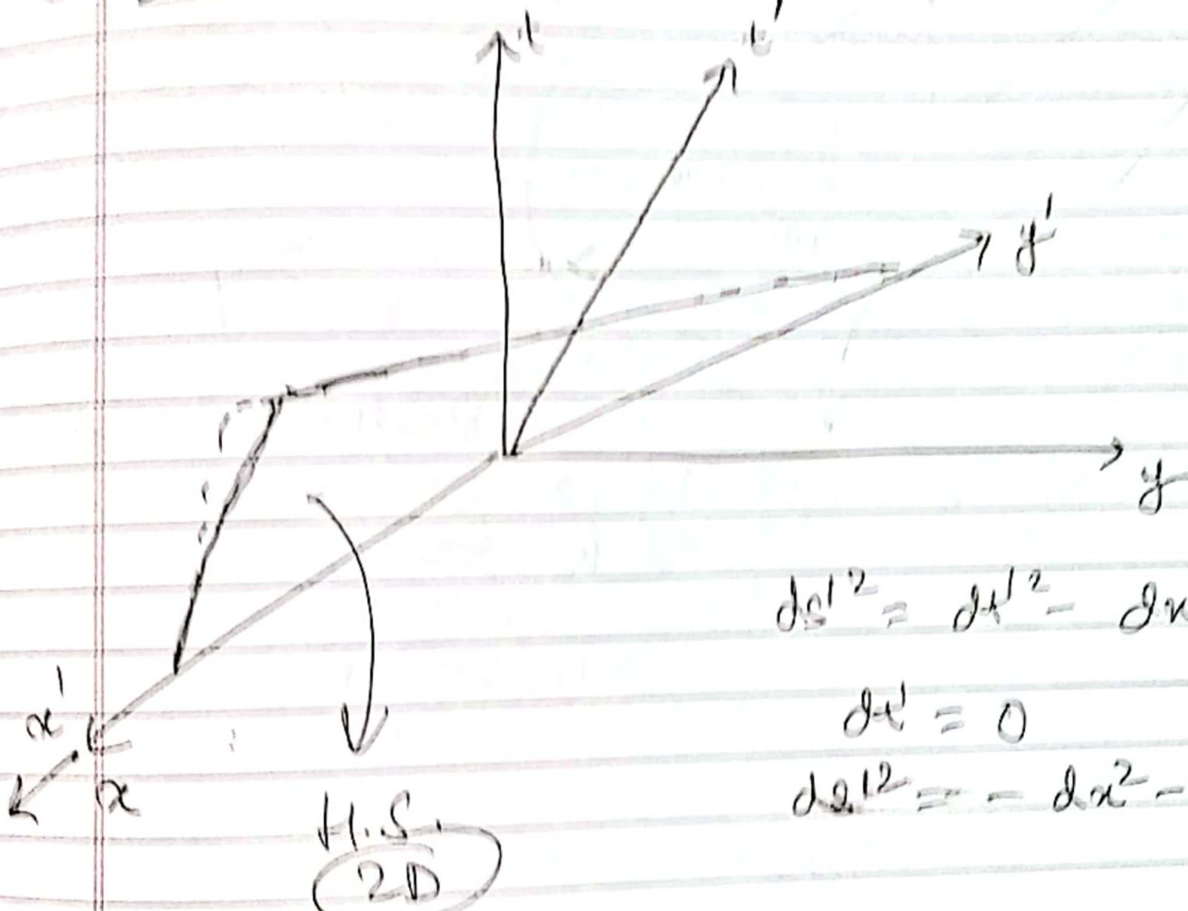


On H.S. (1) $dt = dy$

On H.S. (2) $dy = -dt$

for Both $ds^2\Big|_{hyp.} = -dx^2$

(11) for Timelike

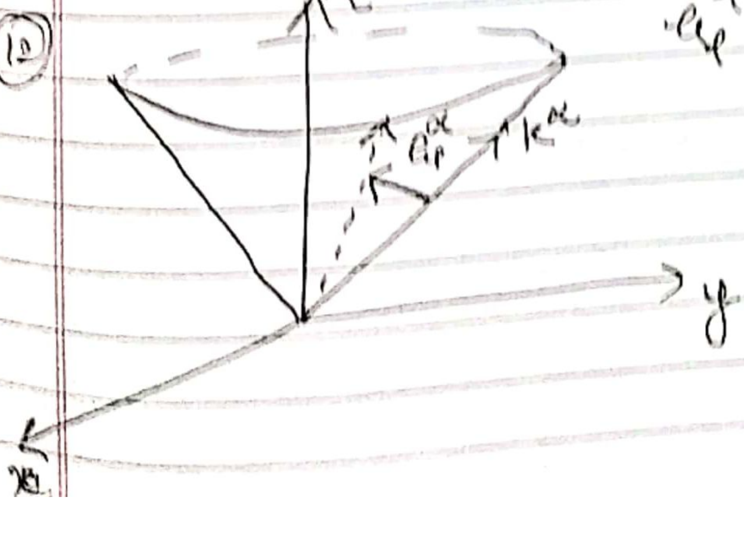


$$ds'^2 = dt'^2 - dx'^2 - dy'^2$$

$$dt' = 0$$

$$ds'^2 = -dx^2 - dy'^2$$

(12)

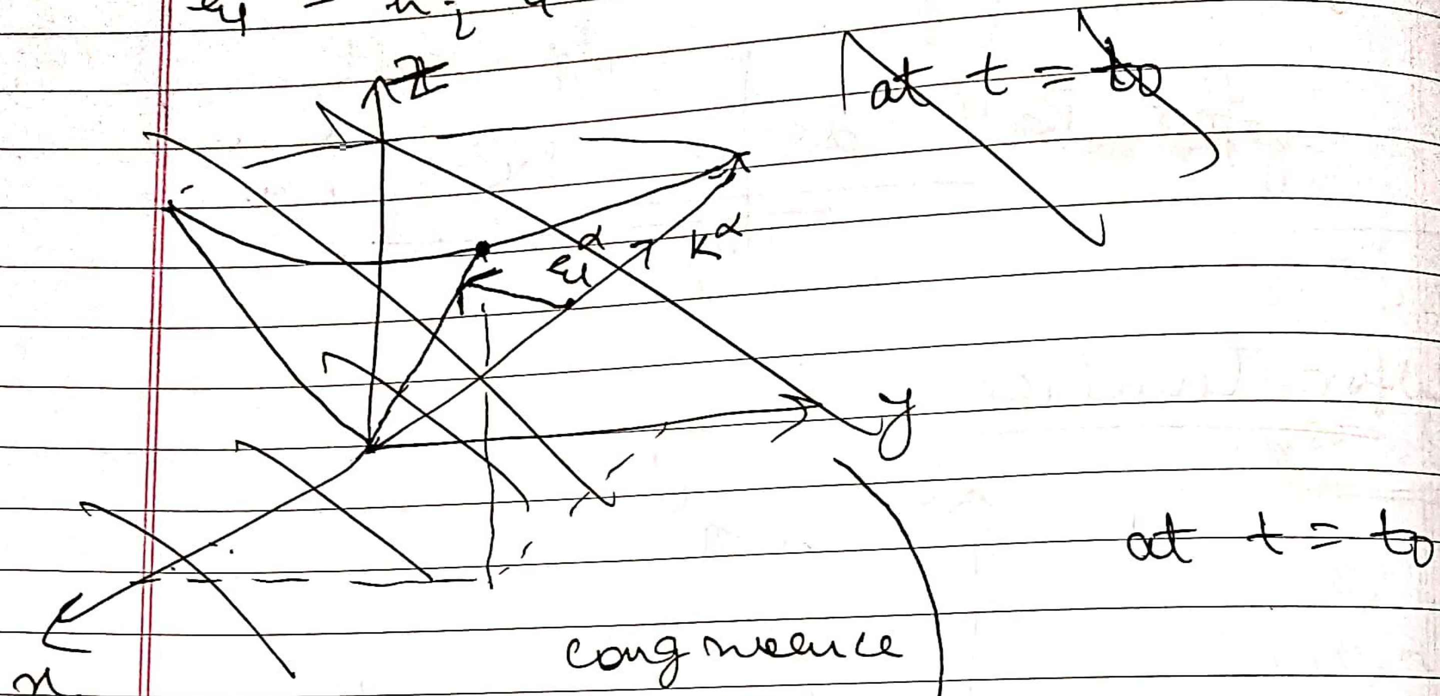


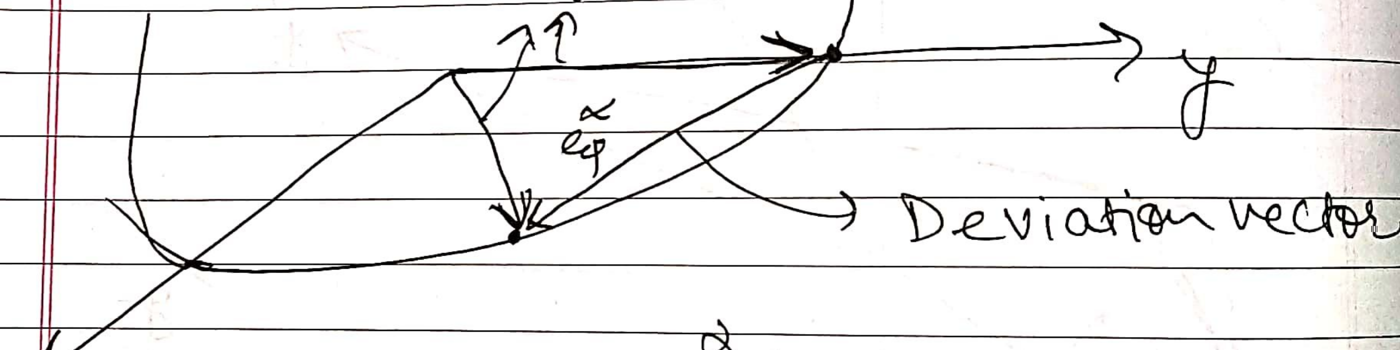
$e_\varphi^\alpha k_\alpha = 0$ (e_φ^α can be in any direction ⊥r or ∥ Both will work out.

as seen ← Not as Def.

∴ e_φ^α Not Necessarily ⊥r to k^α as seen)

$$\tilde{\xi}^\alpha_4 = h^\alpha_i \, \xi^i_4 = \text{Transverse Dev. Vector}$$





$$\xi^\alpha_4 \, k_\alpha = 0$$

(13) from (9)
We have $k^\beta \nabla_\beta \tilde{e}_1^\alpha$ which has comp. in k direction
But we need to kill this component & get into transverse direction.

(14) Transverse Rel. velocity.

$$\left(k^\beta \nabla_\beta \tilde{e}_1^\alpha\right) \equiv \underbrace{h^\alpha_i \left(k^\beta \nabla_\beta \tilde{e}_1^i\right)}_{\text{Purely Transverse}}$$

$$= h^\alpha_i \left(e_1^\beta h^i_r \left(\nabla_\beta k^r\right) - k^i \left(k^\beta e_1^r \nabla_\beta N_r\right)\right)$$

$$= h^\alpha_i h^i_r e_1^\beta \left(\nabla_\beta k^r\right) - \cancel{\left(h^\alpha_i k^i\right)} \left(k^\beta e_{1r} \nabla_\beta N^r\right)$$

$$\boxed{= h^\alpha_r e_1^\beta \nabla_\beta k^r}$$

from 2–7 (37)

$$\tilde{e}_1^\alpha = e_1^\alpha + k^\alpha \left(N_\beta e_1^\beta\right)$$

$$= h^\alpha_r \left(\tilde{e}_1^\beta - k^\beta \left(N_i e_1^i\right)\right) \nabla_\beta k^r$$

$$= h^\alpha_r \tilde{e}_1^\beta \nabla_\beta k^r - \cancel{h^\alpha_r k^\beta \nabla_\beta k^r}$$

$$\boxed{= h^\alpha_r \tilde{e}_1^\beta \nabla_\beta k^r}$$

as $k^\beta B^r_\beta = 0$

$B_{\alpha\beta} K^\beta = 0$

To make vector Transverse $\quad h_{\alpha\beta} \xi_i^\beta = \tilde{\xi}_\beta$

To make Tensor Transverse $\quad h^i_\alpha h^j_\beta B^{\alpha\beta} = \tilde{B}^{ij}$

fully projected B

(15) Now
we know

$$\tilde{\xi}_i^\alpha = \xi_i^\alpha + K^\alpha (N_\beta \xi_i^\beta)$$

is obtained
from

$$\tilde{\xi}_i^\alpha = h^\alpha_\beta \xi_i^\beta$$
$$= h^\alpha_i h^i_\beta \xi_i^\beta$$

But $h^i_\beta \xi_i^\beta = \tilde{\xi}_i^i$

$$\therefore \tilde{\xi}_i^\alpha = h^\alpha_i \tilde{\xi}_i^i$$

(16) $\therefore$ from (14)

$$\left(\widetilde{K^\beta \nabla_\beta \tilde{\xi}_i^\alpha}\right) = h^\alpha_\gamma \tilde{\xi}_i^\beta (\nabla_\beta K^\gamma)$$
$$= h^\alpha_\gamma h^\beta_i \tilde{\xi}_i^i (\nabla_\beta K^\gamma)$$
$$= h^\alpha_\gamma h^\beta_i \tilde{\xi}_i^i B^\gamma_\beta$$
$$= \underbrace{h^\alpha_\gamma h^\beta_i B^\gamma_\beta}\, \tilde{\xi}_i^i$$

Purely Transverse q velocity

$$= \tilde{B}^\alpha_i \tilde{\xi}_i^i$$

fully projected

(17) $$\left(\widetilde{K^\beta \nabla_\beta \tilde{\xi}_i^\alpha}\right) = \tilde{B}^\alpha_i \tilde{\xi}_i^i$$

$$\tilde{B}^\alpha_i = h^\alpha_\gamma h^\beta_i B^\gamma_\beta$$

(18) Now Transverse B in irreducible form.

$$\tilde{B}_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{2} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

(19) $\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta}$ as $h^{\alpha\beta}\sigma_{\alpha\beta} = h^{\alpha\beta}\omega_{\alpha\beta} = 0$

Expansion scalar

from Def. of $h^{\alpha\beta} = g^{\alpha\beta} - K^{\alpha}N^{\beta} - N^{\alpha}K^{\beta}$

But Now $\tilde{B}_{\alpha\beta}K^{\beta} = \tilde{B}_{\alpha\beta}N^{\beta} = 0$

Earlier $B_{\alpha\beta}K^{\beta} = 0$

But $B_{\alpha\beta}N^{\beta} \neq 0$

$\therefore \quad h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta}$

(20) $\theta = \frac{1}{\delta A}\frac{d\,\delta A}{d\lambda}$ $\quad \delta A$ = Cross sectional Area

$\theta = \nabla_i K^i$ (Proof (23))

(21) Shear $\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{h_{\alpha\beta}}{2}\theta$

Rotation $\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

(22) from (17)

$$\tilde{B}^\alpha_i = h^\alpha_r \, h^\beta_i \, B^r_\beta$$

$$= \left(\delta^\alpha_r - K^\alpha N_r - N^\alpha K_r\right)\left(\delta^\beta_i - K^\beta N_i - N^\beta K_i\right)B^r_\beta$$

$$= \left(\qquad\qquad\right)\left(B^r_i - \left(N^\beta B^r_\beta\right)K_i\right)$$

$$= \left(B^\alpha_i - K^\alpha\left(B^r_i N_r\right) - N^\alpha \cancel{K_r B^r_i}\right)$$

$$- \left(\left(N^\beta B^\alpha_\beta\right)K_i - K^\alpha K_i\left(N_r N^\beta B^r_\beta\right) - N^\alpha K_i\left(N^\beta \cancel{K_r B^r_\beta}\right)\right)$$

$$\tilde{B}^\alpha_i = B^\alpha_i - \underbrace{K^\alpha\left(B^r_i N_r\right) - \left(N^\beta B^\alpha_\beta\right)K_i}_{} + \underbrace{K^\alpha K_i\left(N_r N^\beta B^r_\beta\right)}_{}$$

No components along K & N By construction

} kills of N components

} has N comp.

(23) By (19) $\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta}$

$$g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\left(B_{i\alpha} - K_\alpha\left(B^r_i N_r\right) - \left(N^\beta B_{\beta\alpha}\right)K_i + K_\alpha K_i\left(N_r N^\beta B^r_\beta\right)\right)$$

$$g^{\alpha i}\tilde{B}_{\alpha i} = g^{i\alpha}B_{i\alpha} = g^{i\alpha}\nabla_i K_\alpha$$

$$\Rightarrow \nabla_i K^i = \theta$$

(24) As $N_\alpha N^\alpha = 0$ } 2 conditions & we have
$K_\alpha N^\alpha = -1$ } to obtain 4 components

$\therefore$ under constrained Many solutions $\exists$

$\therefore h_{\alpha\beta} = g_{\alpha\beta} - K_\alpha N_\beta - K_\beta N_\alpha$ is Not Unique

$\therefore \tilde{B}_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{2} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$

$\hookrightarrow$ $\boxed{\text{Not Unique}}$

But $\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = \nabla_i K^i$

Independent of $h^{\alpha\beta}$ & N_α

$\therefore$ Expansion is independent of Null Direction & is Unique

(25) Shear & Rotation Depends on the choice of Null Direction. $\rightarrow$ Why?

(26) Frobenius Th.

Th: Congruence is Hyp. "Orth" iff $\omega_{\alpha\beta} = 0$
(surface forming)

Proof: Let Cong. be Hyp. "Orth"

$\therefore K_\alpha = A\partial_\alpha \phi$
function

$\phi = \text{const}$

why A has to be a function? why not Const.

$B_{\alpha\beta} = \nabla_\beta K_\alpha = A\nabla_\beta\nabla_\alpha\phi + (\nabla_\alpha\phi)(\nabla_\beta A)$

$= A\nabla_\beta\nabla_\alpha\phi + \frac{K_\alpha}{A}(\nabla_\beta A)$

We are interested in $\tilde{B}_{\alpha\beta}$ as in (22)

$\therefore$ we need $B_{\alpha\beta} N^\beta$

$$\therefore B_{\alpha\beta} N^\beta = \nabla_\beta (A \nabla_\alpha \phi) N^\beta$$
$$= A(\nabla_\beta \nabla_\alpha \phi) N^\beta + (\nabla_\beta A)(\nabla_\alpha \phi) N^\beta$$
$$\boxed{= A(\nabla_\beta \nabla_\alpha \phi) N^\beta + (\nabla_\beta A) \frac{K_\alpha}{A} N^\beta}$$

$$N^\alpha B_{\alpha\beta} = N^\alpha (\nabla_\beta K_\alpha)$$
$$= N^\alpha \nabla_\beta (A \nabla_\alpha \phi)$$
$$= N^\alpha A (\nabla_\beta \nabla_\alpha \phi) + N^\alpha (\nabla_\beta A)(\nabla_\alpha \phi)$$

But $N^\alpha K_\alpha = -1$

$$\boxed{\therefore N^\alpha B_{\alpha\beta} = N^\alpha A (\nabla_\beta \nabla_\alpha \phi) - \frac{\nabla_\beta A}{A}}$$

$$B_{\alpha\beta} N^\alpha N^\beta = (B_{\alpha\beta} N^\beta) N^\alpha \quad \text{(as this is a scalar)}$$
$$= \left(A(\nabla_\beta \nabla_\alpha \phi) N^\beta + \frac{(\nabla_\beta A)}{A} K_\alpha N^\beta\right) N^\alpha$$
$$= A(\nabla_\beta \nabla_\alpha \phi) N^\beta N^\alpha + \frac{(\nabla_\beta A) N^\beta K_\alpha N^\alpha}{A}$$
$$= A(\nabla_\beta \nabla_\alpha \phi) N^\beta N^\alpha - N^\beta (\nabla_\beta A)$$

Putting all this in $\tilde{B}_{\alpha\beta}$ (22)

$$\therefore \tilde{B}_{\alpha\beta} = \{ A \nabla_\beta \nabla_\alpha \phi + \frac{k_\alpha}{A} (\nabla_\beta A) - A k^{a}_\alpha \nabla_\beta \nabla$$

Symmetric

α, β exchange I get other term

$$\tilde{B}_{\alpha\beta} = A \nabla_\alpha \nabla_\beta \phi + A k_\alpha \nabla_i \nabla_\beta \phi N^i + A k_\beta \nabla_\alpha \nabla_i \phi N^i + A k_\alpha k_\beta \nabla_i \nabla_j \phi N^i N^j$$

Symmetric

Symmetric

$\therefore \tilde{B}_{\alpha\beta}$ is completely symm. $\boxed{\therefore \omega_{\alpha\beta} = 0}$

Frob. Theorem

(27) Hyp. Orth $\Longleftrightarrow$ $\omega_{\alpha\beta} = 0$ for Null Geodesic

(28) ~~Theorem also holds True for Null lines which are Not Geod.~~

(29) we know k^α, pick N^α, Calculate $\omega_{\alpha\beta} = 0$.

$\therefore$ we know $k^\alpha = \partial_\alpha \phi$

find ϕ. solve D.E

(30) Raychaudhuri Eqn

$$\frac{d\theta}{d\lambda} = \tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}k^\alpha k^\beta$$

(Same Calculation as in Timelike)

From (22): $\boxed{B^{\alpha\beta}B_{\beta\alpha} = \tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha}}$

$$\therefore \frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} + \sigma^{\alpha\beta}\sigma_{\beta\alpha} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta$$

(31) Focusing Theorem

In Regards to Black hole we will be given Null Surfaces

Null Surface is always s.t. Null geod. are congruence moving along it.

If cong. is HyperSurf Orth $\Rightarrow$ $\omega_{\alpha\beta} = 0$

If $R_{\alpha\beta}k^\alpha k^\beta \geq 0$ (Null Energy Condn)

If EFE $\quad R_{\alpha\beta}k^\alpha k^\beta = 8\pi\left(T_{\alpha\beta} - \frac{T g_{\alpha\beta}}{2}\right)k^\alpha k^\beta$

$R_{\alpha\beta}k^\alpha k^\beta = 8\pi T_{\alpha\beta}k^\alpha k^\beta$

Hyp... Normal ... Null Surface ... If Every ... moving ...

$\Rightarrow$ R...

$\therefore$ ...

(32) Doubt: As we ...

But

$\therefore$ ...

$\therefore$ ...

(33) If ...

here if Null geo... are given it's necessary ... Null Surface ...

$\Rightarrow$ But if Null Surface is there then Null geod. $\Rightarrow$ Sin...

But ...

$\therefore$ If for ...

classmate
Date
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Hyper Surface $\phi = \text{const}$
Normal $n_\alpha \propto \partial_\alpha \phi$
Null Surface : $n_\alpha n^\alpha = g_{\alpha\beta} \partial^\alpha \phi \partial^\beta \phi = 0$

if Energy Density as measured by observer moving with speed of light, is positive

$\Rightarrow R_{\alpha\beta} k^\alpha k^\beta \geq 0$

$\therefore \frac{d\theta}{d\lambda} \leq 0$

(32) Doubt As we know θ is ind of N

σ, ω are not Unique

But $\sigma_{\alpha\beta}\sigma^{\alpha\beta}$, $\omega_{\alpha\beta}\omega^{\beta\alpha}$ are Unique.

$\Rightarrow \therefore \frac{d\theta}{d\lambda}$ is ind. of N

$\therefore$ Focusing Theorem is ind. of N

(33) if $\sigma_{\alpha\beta}\sigma^{\alpha\beta} = 0$ & σ Being 2 Dimension Spatial

here if Null geod are given it is not necessary a Null surface would be.
$\Rightarrow$ But if Null surface is there then Null geod.

$\therefore \sigma_{\alpha\beta} = 0$

But $\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ is Unique

$\therefore \sigma_{\alpha\beta} = 0$ (Unique).

Similar for $\omega_{\alpha\beta}\omega^{\alpha\beta}$

$\therefore$ ~~Stat~~ Frob. Theorem is Ind. N
if for some N, $\omega_{\alpha\beta} = 0$ then it zero for any N

2.5 : 6, 8
3.13 : 1, 2

9 Self

L-9

① Numerical Relativity originates from initial value data from hypersurfaces.

② Description
Integration
Intrinsic & Extrinsic Curvature
Initial Value Problem.

③ 2 ways to describe HyperSurface in 4D
① By Algebraic eqn
② By Parametric eqn

Eg. 2-sphere in 3D

$\Rightarrow \phi(x, y, z) = 0$

$x^2 + y^2 + z^2 - R^2 = 0$

How to embed 1-D circle in 3D?

$$\Rightarrow \left.\begin{aligned} f_1(\theta, \phi) &= x \\ f_2(\theta, \phi) &= y \\ f_3(\theta, \phi) &= z \end{aligned}\right\} \text{Parametric eqn}$$

3 fn & 2 parameters

Similarly for line in 2D

$\Rightarrow \phi(x, y) = 0$

$mx + y + c = 0$

$$\Rightarrow \left.\begin{aligned} f_1(t) &= x \\ f_2(t) &= y \end{aligned}\right\} \text{Param.}$$

2 fn & 1 parameter

See ch-1 (80)

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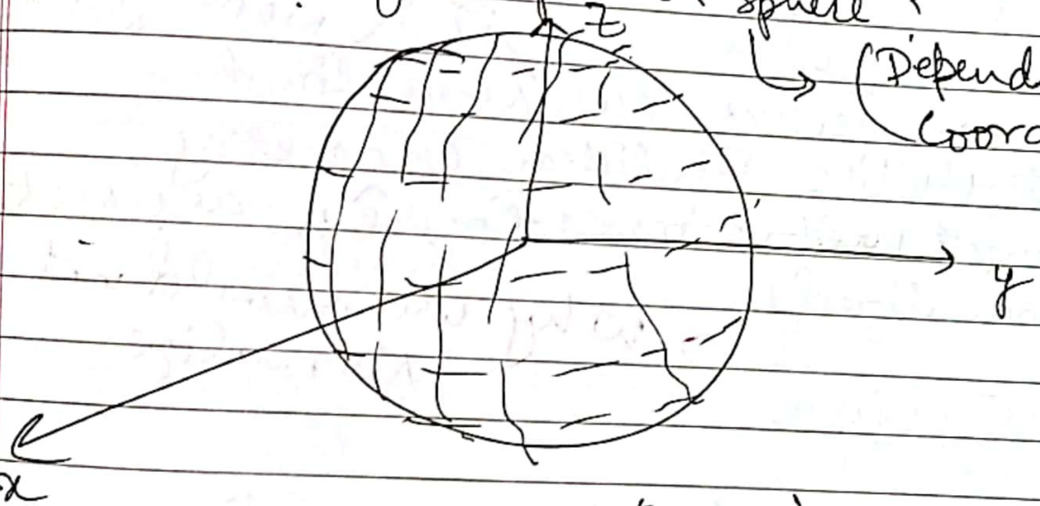
(4) 2-Sphere in 3D

Parametric eq^n

$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \text{Param.}$$

What parameters (θ, ϕ) do I use depends on ~~use~~? grid put on sphere? → (Depends on Intrinsic Coordinates)



(x, y, z)
Spacetime Coordinates need not be same as the coordinates we used on Hypersurf. ← (θ, ϕ)

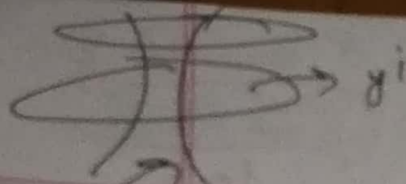
Algebraic description is good to get Normal

(5) $\vec{n} \propto \vec{\nabla}\phi$

$$\vec{n} \propto (2x, 2y, 2z)$$

Normalize

$$\vec{n} = (x/R, y/R, z/R)$$



$$x^\alpha = x^\alpha(\tau, y^i)$$

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial \tau}\right)_{y^i} d\tau + \left(\frac{\partial x^\alpha}{\partial y^i}\right)_\tau dy^i \rightarrow dx^\alpha\Big|_{H/S} = \left(\frac{\partial x^\alpha}{\partial y^i}\right)_\tau$$

(6) Parametric eqn is good to get Tangent Vectors on Surface

$$d\vec{x} = \frac{\partial \vec{x}}{\partial \theta} d\theta + \frac{\partial \vec{x}}{\partial \phi} d\phi$$

But also

$$d\vec{x} = d\theta\, \hat{e}_\theta + d\phi\, \hat{e}_\phi$$

$$\hat{e}_\theta = \frac{\partial \vec{x}}{\partial \theta} = (R\, c\theta\, c\phi,\ R\, c\theta\, s\phi,\ -R\, s\theta)$$

$$\hat{e}_\phi = \frac{\partial \vec{x}}{\partial \phi} = (-R\, s\theta\, s\phi,\ R\, s\theta\, c\phi,\ 0)$$

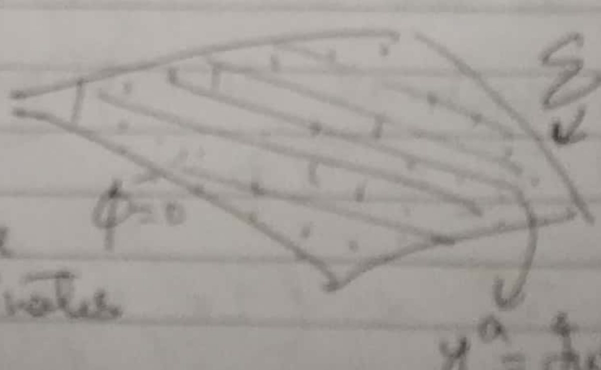
(7) Normal unit vectors are Normalized → for Null case Normals are Not Normalized

But Unlike Euclidean Geometry Tangent ~~unit~~ vectors $\hat{e}_\theta$, $\hat{e}_\phi$ are not Normalized. → why we should not Normalize

(8) Hyper Surface

3D submanifold of our spacetime manifold. Therefore it should have all the smoothness properties

$$\phi(x^\alpha) = 0$$ for all H/S to be



x^α = Spacetime Coordinates

$$x^\alpha = f^\alpha(y^a)$$ - Parametric Description → of H.S

y^a = Intrinsic Coord.

Unit Normal is not defined for Null Surface.
In Null Case $n_\alpha = \partial_\alpha \phi$

(9) for S/T like hypersurface.

$n_\alpha \propto \partial_\alpha \phi \Rightarrow$ Normalize.
Normal can point in/out of the surface.
Convention: $\to n_\alpha$ points in the direction of increasing ϕ if ϕ is spacelike.
$\to$ if there is inside & outside to the surface, pick n_α s.t. it $\uparrow$ from inside out.
In 3D $\vec{n} \cdot \vec{\nabla}\phi > 0$

Unit Normal can be introduced if H.S is Not Null.

(10) $$n_\alpha = \frac{\epsilon\, \partial_\alpha \phi}{|g_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi|^{1/2}} \qquad \epsilon \equiv n_\alpha n^\alpha = \begin{cases} 1 & \text{if } \Sigma \text{ spacelike} \\ -1 & \text{if } \Sigma \text{ timelike} \end{cases}$$

can the surface be timelike at one pt. & stat other?

How can I know if the surface is spacelike/timelike? $\to$ compare with flat spacetime

(11) Tangent vectors

$$e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a} \quad \text{as} \quad dx^\alpha\Big|_{\text{along H.S.}} = \frac{\partial x^\alpha}{\partial y^a} dy^a = e^\alpha_a dy^a$$

(12) $$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b} dy^a dy^b$$

Pull back of the metric to H.S.

Induced metric which changes coordinates diff. to intervals.

$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a e^\beta_b$$

(13) we have 2 coord. system
(1) Global coord. system
(2) Intrinsic coord. system of H.S.
There is a diff. B/w Tensor in Spacetime & Tensor in Intrinsic coord. syst.

(14) $h_{ab} = g_{\alpha\beta}\, e^{\alpha}_{a}\, e^{\beta}_{b}$ — Scalar Relative to Transfn of x^α

→ Tensor Relative to transf of y^a

$\boxed{\text{How?}}$

How?

Completeness Relation

(15) Now we have Basis vectors; n^α, e^α_a.

$$g_{\alpha\beta} = \epsilon\, n_\alpha n_\beta + h_{\alpha\beta}$$

where $h_{\alpha\beta} n^\alpha = 0$

(Similar to Longmuir?)

$$\Rightarrow g_{\alpha\beta} n^\alpha n^\beta = \epsilon (n_\alpha n_\beta)(n^\alpha n^\beta) + \cancel{h_{\alpha\beta} n^\alpha n^\beta}$$

$$= \epsilon\, \underbrace{(n_\alpha n^\alpha)}_{\epsilon}\, \underbrace{(n_\beta n^\beta)}_{\epsilon}$$

$$\underline{g_{\alpha\beta} n^\alpha n^\beta = \epsilon}$$

~~But~~ h_{ab} is Scalar in Spacetime? But $h_{\alpha\beta}$ is Tensor in ST.

(16) As $h_{\alpha\beta}$ is Tangent to H.S.

∴ They Can be decomposed int e^α_a.

$$\therefore\ h^{\alpha\beta} = \bigcirc\, e^\alpha_a\, e^\beta_b$$

Let $h^{\alpha\beta} = A^{ab}\, e^\alpha_a\, e^\beta_b$

(17) Claim: $A^{ab} = h^{ab}$

Definition: $h^{ab} h_{bc} = \delta^a_c$ how?

Proof: $h_{mn} = g_{\alpha\beta} e^\alpha_m e^\beta_n$

$= (\epsilon n_\alpha n_\beta + h_{\alpha\beta}) e^\alpha_m e^\beta_n$

We know $n_\alpha e^\alpha_m = 0$

$\therefore h_{mn} = h_{\alpha\beta} e^\alpha_m e^\beta_n = g_{\alpha\gamma} g_{\beta\delta} h^{\gamma\delta} e^\alpha_m e^\beta_n$

$= h^{\gamma\delta} e_{\gamma m} e_{\delta n} = h^{\alpha\beta} e_{\alpha m} e_{\beta n}$

$\boxed{\cancel{As\ e^\alpha_m = h^{\alpha\beta} e_{\beta m}}}$

$h_{mn} = (A^{ab} e^\alpha_a e^\beta_b) e_{\alpha m} e_{\beta n}$

$= A^{ab} (e^\alpha_a e_{\alpha m})(e^\beta_b e_{\beta n})$

$= A^{ab} (g_{\alpha\beta} e^\alpha_a e^\beta_m)(g_{\beta\alpha} e^\beta_b e^\alpha_n)$

$h_{mn} = A^{ab} h_{am} h_{bn}$

$h_{mn} = (A^{ab} h_{am}) h_{bn}$

$\therefore A^{ab} h_{am} = \delta^b_m \Rightarrow A^{ab}$ is inverse of h_{am}

$A^{ab} = h^{ab} \Rightarrow \boxed{h^{\alpha\beta} = A^{ab} e^\alpha_a e^\beta_b}$

(18) Null like

(1) For Timeli/ Spacelike

$$n_\alpha = \frac{\epsilon \partial_\alpha \phi}{\sqrt{|g^{\alpha\beta}\partial_\alpha\phi \, \partial_\beta\phi|}}$$

But in Nullike

$$\partial_\alpha \phi \, \partial^\alpha \phi = 0$$

$\therefore$ Nomal can't be Nomalized
$\therefore$ Normalization is arbitrary

(2) $n_\alpha n^\alpha = 0$ $\therefore$ They are Tangent & Nomal in same time.

Null Case

(19) $\phi(x^\alpha) = 0$; $f^\alpha(y^a) = x^\alpha$

But here $\partial_\alpha \phi$ is Null vector
$\therefore$ cannot obtain Unit Nomal

Nomal

Convention $k_\alpha \equiv + \partial_\alpha \phi$ (ϕ increasing towards future
(k^α is future pointing)

$k^\alpha k_\alpha = 0$

$\therefore$ k^α is also Tangent to H.S.

(20)

k^α is Tangent to Null curves in Σ.

(21) When we are given Null H.S.
we are also given k^α which are Tangent in Σ
∴ we are also given Null curves in Σ.

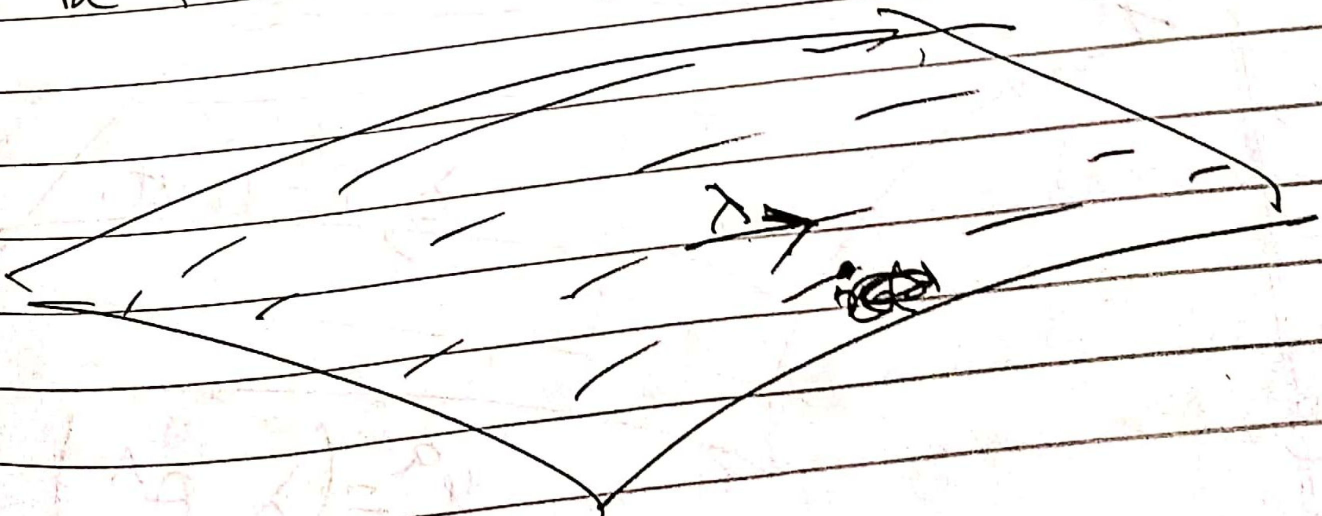
These Null Curves are Geodesics.

&
These Null Geodesics are congruence of H.S. Orth. to Null Geod.

Def: These Null Geod. are called null Generators

(22) we will pick Intrinsic coord. y^a that are adapted to Network of null curves.

λ be the Running parameter on each curve

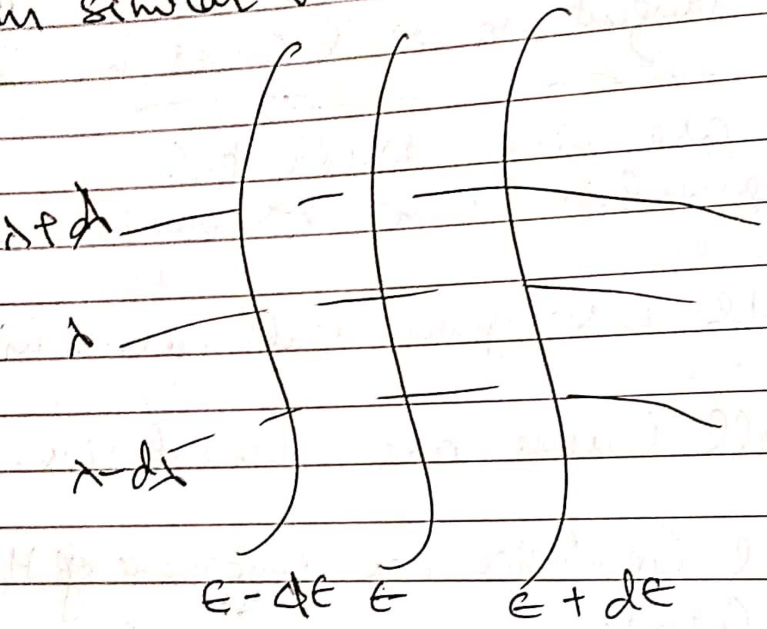


(23) Let $y^1 \equiv \lambda$ from (22)

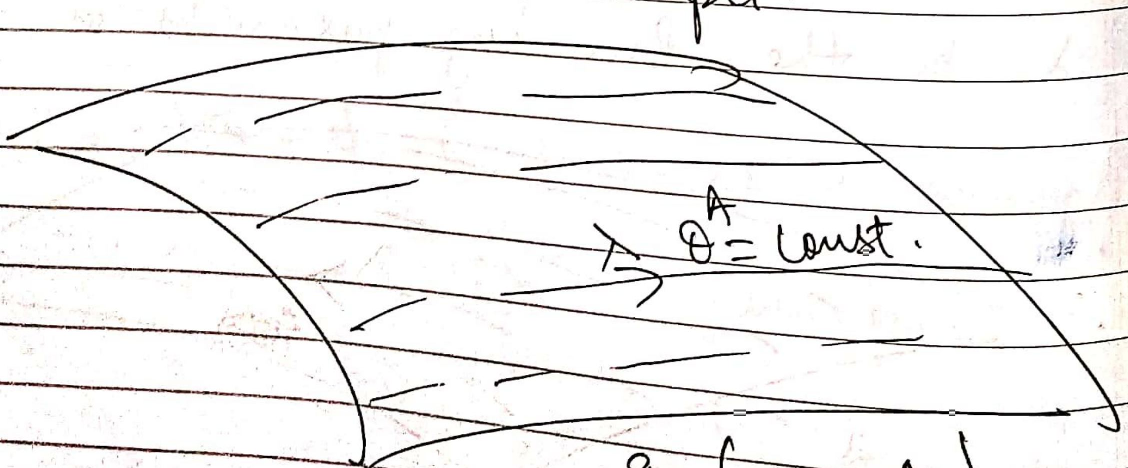
& Remaining 2 coord. will be constant on each of the null curves.

$$y^2, y^3 \equiv \theta^2, \theta^3 \to \theta^A : \text{constant on each null curve}$$

In similar dein



This is what we mean by adapted coord. syst.
By this coord. syst. Description of null curves becomes simple.



$$y^a = (\lambda, \theta^A)$$

why? Null

(24) $e^\alpha_1 = \dfrac{\partial x^\alpha}{\partial y^1} = \left(\dfrac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A = \text{const.}} = k^\alpha$

Surface

Spacelike

$e^\alpha_A = \left(\dfrac{\partial x^\alpha}{\partial \theta^A}\right)_{\lambda = \text{const.}}$

$\boxed{\text{In timelike case Basis vectors } e_i \text{ are ~~timelike~~ spacelike.}}$

(25) Condn to impose : $\boxed{k_\alpha e^\alpha_A = 0}$

Restricts the freedom to choose θ^A

This comes from construction itself? Yes

(26) If we take these coord. int. system then all the subtelities of Null curves go away.

(27) Null curves we are talking about are Null geodesics.

Proof: $a^\alpha = k^\beta \nabla_\beta k^\alpha$ either $a^\alpha = 0$ or $a^\alpha \propto k^\alpha$

$a^\alpha = \dfrac{Dk^\alpha}{d\lambda}$ k^α is along H.S. But $\dfrac{Dk^\alpha}{d\lambda}$ can have comp. $\perp^r$ to H.S.

see (28)

$\therefore a^\alpha = ck^\alpha + a^A e^\alpha_A + bN^\alpha$ → see (28)

$N^\alpha \perp^r$ to H.S.

$k_\alpha = -\partial_\alpha \phi$

{ b is Comp. of a^α along N^α }
{ b is Comp. of a^α along k^α }

another weird thing about Null: we can't say $a^\alpha = ck^\alpha + dN^\alpha$ then c is comp of k^α as in L. Alg → Diff

$$a_\alpha k^\alpha = -b = (k^\beta \nabla_\beta k^\alpha) k^\alpha = \frac{k^\beta}{2} \nabla_\beta (k^\alpha k_\alpha)$$

$\therefore a^\alpha = ck^\alpha + d^A e^\alpha_A$ only along H.S.

$\frac{D(k^\alpha k_\alpha)}{d\lambda}$ along k^α with param λ

In timelike case we have $\{n_\alpha, e^\alpha_a\}$ 4 Basis vectors

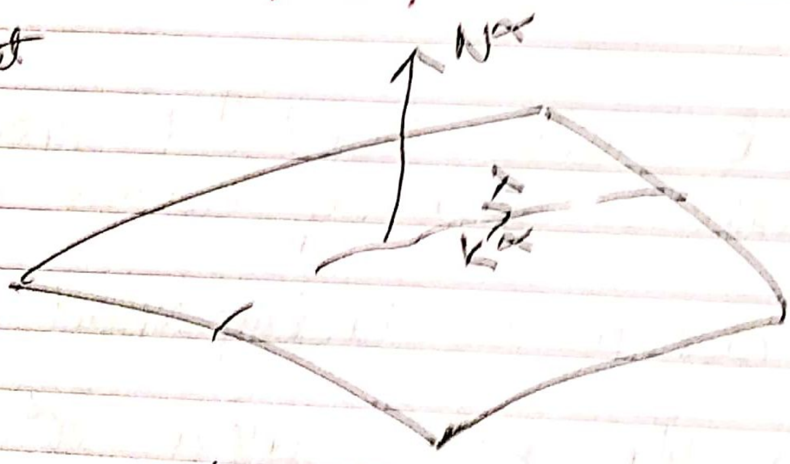
But in null like case ours $e^\alpha_1 = k^\alpha$ (coincides)

$\therefore$ we have 3 Basis vectors only.

$\therefore$ I need 1 more basis

Basis Definition

let



N^α = Null vector going away from H.S.

Doubt How is $N^a e^A_a = 0$ & $N^a k_a = 1$

s.t. 4 cond

$$\boxed{\begin{aligned} N^\alpha N_\alpha &= 0 \\ N_\alpha k^\alpha &= +1 \\ N_\alpha e^\alpha_A &= 0 \end{aligned}}$$

How? (As Before)

$A = 1, 2$

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These set of condⁿ produces unique solⁿ.
Now we have full 4 Basis vectors.

(2)
$$a_\alpha e^\alpha_B = a^A \underbrace{\left(e_{\alpha A} e^\alpha_B\right)}_{\neq 0}$$

$$= a^A \left(g_{\alpha\beta} e^\beta_A e^\alpha_B\right)$$

$$= a^A h_{AB}$$

$$= \left(k^\beta \nabla_\beta k_\alpha\right) e^\alpha_B$$

But $k_\alpha = -\partial_\alpha \phi$

$$\nabla_\beta k_\alpha = -\nabla_\beta \nabla_\alpha \phi = -\partial_\beta \partial_\alpha \phi$$

$$= -\partial_\alpha \partial_\beta \phi = -\nabla_\alpha \nabla_\beta \phi$$

$$\nabla_\beta k_\alpha = \nabla_\alpha k_\beta \Rightarrow \cancel{\nabla_\beta k^\alpha = \nabla^\alpha k_\beta}$$

$$\cancel{\therefore a_\alpha e^\alpha_B = k^\beta \nabla^\alpha k_\beta e^\alpha_B}$$

$$a_\alpha e^\alpha_B = \left(k^\beta \nabla_\alpha k_\beta\right) e^\alpha_B = \frac{\nabla_\alpha}{2}\left(k_\beta k^\beta\right) e^\alpha_B$$

$$= 0 \qquad \frac{D(k^\beta k_\beta)}{dr} = 0 \text{ along } e^\alpha_B \text{ with param. } r$$

$$\therefore a^\alpha = c k^\alpha$$

$$a_\alpha N^\alpha = -c = k^\beta \left(\nabla_\beta k_\alpha\right) N^\alpha = k^\beta \left(\nabla_\alpha k_\beta\right) N^\alpha$$

$$= \nabla_\alpha \left(k_\beta k^\beta\right) N^\alpha \neq 0$$

As $k^\alpha k_\alpha = 0$ along the Surf.

∴ Any Cov. Derivative of $k^\alpha k_\alpha$ along the Curve tang to H.Surf. $= 0$

(30) Now as

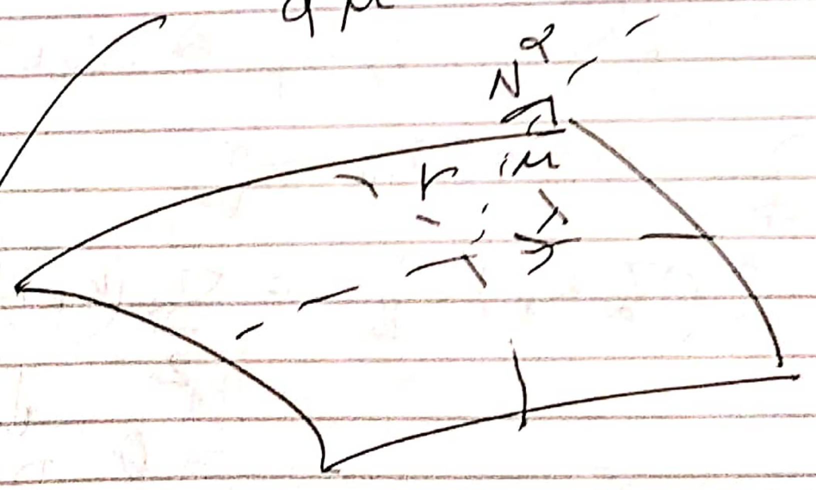
$$u^\alpha u_\alpha = -1 \quad \text{all along the spacetime}$$

$$\frac{D(u^\alpha u_\alpha)}{d\lambda} = 0$$

Cov. Deriv. along any Curve in Spacetime $= 0$

But $k^\alpha k_\alpha = 0$ only along H.S.

∴ $\frac{D(k^\alpha k_\alpha)}{d\mu} \neq 0$ along N^α



Conditions when this will be zero?

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(8) $k^{\beta}\nabla_{\beta}k^{\alpha} = ck^{\alpha}$

↳ Geod. Eq^n

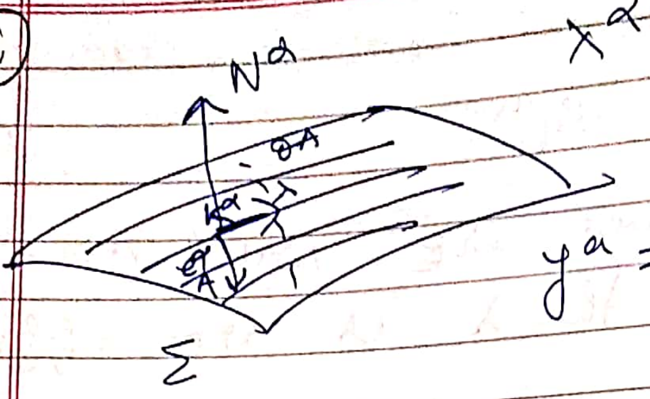
∴ Null curves are Null Geod.

& if $c \neq 0$ then λ is Non affine

$c = 0$ then λ is affine

L-10

(1)



$$y^a = (\lambda, \theta^A)$$

$$\left(N^\alpha, k^\alpha, e^A_\alpha\right)$$

Decomposition of metric

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \lambda} d\lambda + \frac{\partial x^\alpha}{\partial \theta^A} d\theta^A$$

$$= k^\alpha d\lambda + e^\alpha_A d\theta^A$$

$$ds^2 = g_{\alpha\beta}\left(k^\alpha d\lambda + e^\alpha_A d\theta^A\right)\left(k^\beta d\lambda + e^\beta_B d\theta^B\right)$$

$$= g_{\alpha\beta}\, e^\alpha_A e^\beta_B\, d\theta^A d\theta^B$$

$$= \sigma_{AB}\, d\theta^A d\theta^B$$

$\sigma_{AB} \rightarrow$ Induced metric (2D)

Metric is degenerate

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha_A e^\beta_B$$

see ch-3 (14)

1 Row & column which produces 0 eigenvalue.

(2) Hyper Surface is 3D
But Transverse Subspace is 2D } Null Case

Hyper Surface is 3D
Trans. Subspace is ... } Non Null Case

How do I know they are 2D or 3D

3D

$$h_{\alpha\beta} = g_{\alpha\beta} - \epsilon n_\alpha n_\beta$$

$$h_{\alpha\beta} = g_{\alpha\beta} - k_\alpha N_\beta - N_\alpha k_\beta .$$

3D

$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a\, e^\beta_b$$

3D

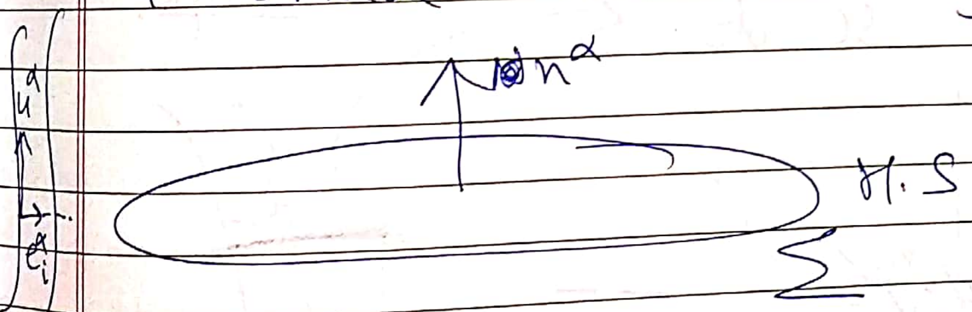
$$\cancel{h_{ab}} = \sigma_{ab} = g_{\alpha\beta}\, e^\alpha_a\, e^\beta_b$$

2D

(3) Non Null

3 Transverse Tangent vectors e^α_i } To the H.S. (3D)

1 Normal N^α



(4) Null Case

2 Transverse ~~and~~ Tangent vector   } To H.S (3D)

1 Tangent vector along curve

1 Normal N^α

5) 2+1 Formalism Null H.S

$y^a = (\lambda, \theta)$

H.S is 2D

$$ds^2 = g_{\alpha\beta}(k^\alpha d\lambda + e^\alpha d\theta)(k^\beta d\lambda + e^\beta d\theta)$$

$$= g_{\alpha\beta} e^\alpha e^\beta d\theta^2$$

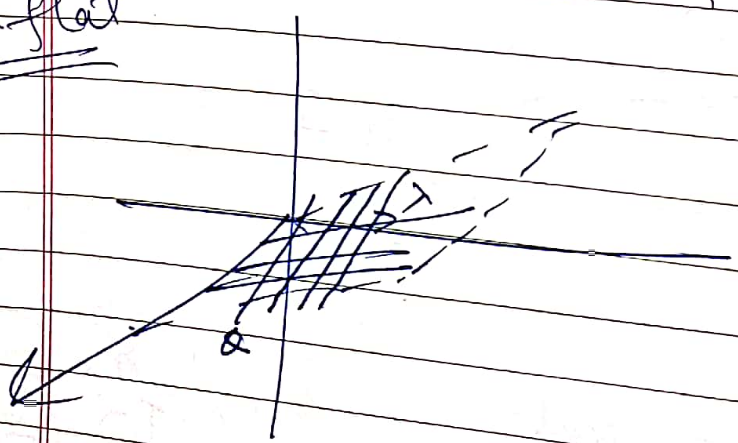
$$ds^2 = l \, d\theta^2$$

$$\left.\begin{aligned} k^\alpha e_\alpha &= 0 \\ N^\alpha e_\alpha &= 0 \\ N^a k_\alpha &= 1 \end{aligned}\right\} \rightarrow \text{How}$$

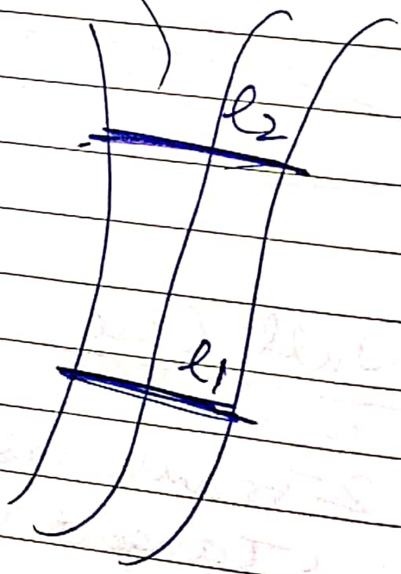
expansion

$$\theta = \frac{1}{\delta l} \frac{d}{d\tau} \delta l$$

in flat



in GR



(6) 2+1 Formalism Non Null HS

(7) $g^{\alpha\beta} = K^{\alpha} N^{\beta} + N^{\alpha} K^{\beta} + \sigma^{ab} e^{\alpha}_{a} e^{\beta}_{b}$

$\sigma^{ab} \sigma_{bc} = \sigma^{a}_{c}$

$g^{\alpha\beta} K_{\alpha} N_{\beta} = 1 \Rightarrow$ To verify this decomposition works

$h^{\alpha\beta} = \sigma^{ab} e^{\alpha}_{a} e^{\beta}_{b}$

(8) Integration

Gauss Theorem 3D flat

$$\int_{V} \underbrace{(\vec{\nabla} \cdot \vec{V})}_{\text{Scalar}} d^3x = \oint_{\partial V} \vec{V} \cdot \vec{n} \underbrace{da}_{\text{Surface Element}}$$

vector

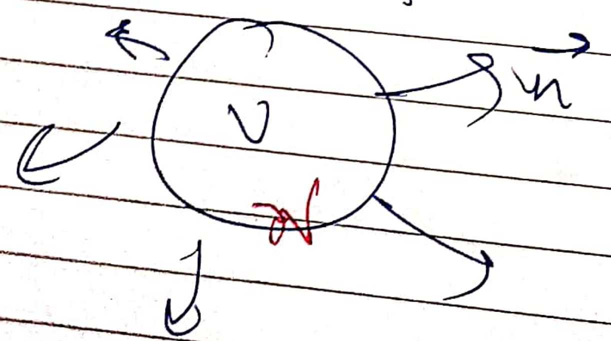
In H.S. we can't integrate vectors as addition of vectors not defined.

∴ scalar

$d\vec{a} = \vec{n}\, da$ direction to surface element

Why area element is vector But volume element is not?

Convention: $\vec{n}$ points outward of the closed surface.



in HyperSurface

(9) Invariant Volume element in 4D

in Spacetime 4D volume $dV = \sqrt{-g}\, d^4x$.

in Hypersurf. 3D Surface element.

Σ : induced metric h_{ab}

Only for Timelike / Spacelike.

Invariant Surf. element in 3D $d\Sigma = \sqrt{|h|}\, d^3y$

Depending on y H.S. is T/S.

see Ch-2 94

for spacelike/ Timelike

(10) Directed Surface element : $n_\mu d\Sigma$

$(n_\mu n^\mu = \epsilon)$

as $\sqrt{-g}$ coz $|g_{ab}|$ is -ve always

how do I know?

(11) $h_{ab} = \partial_a y^{a'} \partial_b y^{b'} h_{a'b'}$

$h = J^2 h'$

But for Null $h_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & \sigma_{ab} \end{pmatrix}$

Metric Becomes Degenerate.

$\therefore \quad h = 0$

(12) $\therefore$ In Null Case $d\Sigma = \sqrt{|h|}\, d^3y$ doesn't generalize.

$n_\mu n^\mu = \epsilon$ in Non Null Case

But in Null Case we don't have Unit Normal vector $\therefore$ This also doesn't generalize.

(13) :- When going to Null case

$$d\Sigma \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0$$
$$n_\mu \longrightarrow \infty$$

$\therefore$ $n_\mu d\Sigma$ gives the notion of some finite no.

(14) In form Language

we will take 4-form to define Volume Elem
3 form to define 3D area element

(15) Directed surf. element in all cases.

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha e_2^\beta e_3^\gamma d^3y$$
$$\epsilon_{\mu\alpha\beta\gamma} = \sqrt{-g}\,[\mu\,\alpha\,\beta\,\gamma]$$

(16) Claim: In space/T case $d\Sigma_\mu$ points in the direction of n_μ normal.

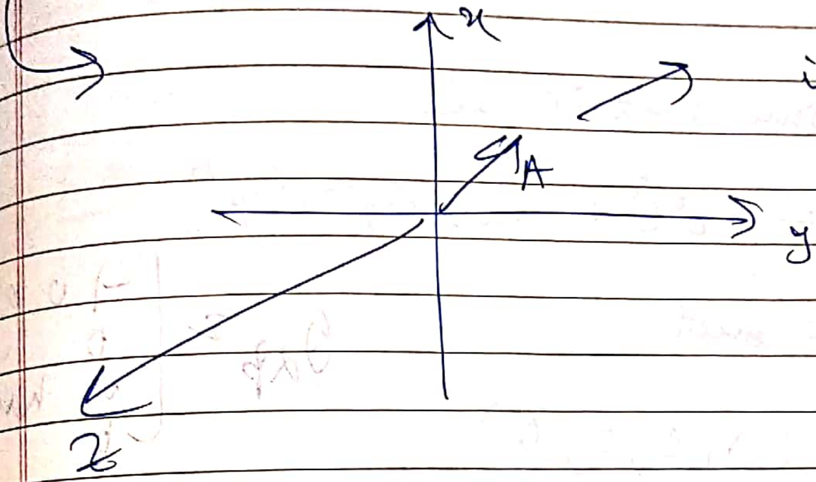
$$d\Sigma_\mu e_2^\mu = e_2^\mu \epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha e_2^\beta e_3^\gamma d^3y$$
$$= \underbrace{\epsilon_{\mu\alpha\beta\gamma}}_{\text{Antsy}} e_1^\alpha \underbrace{e_2^\mu e_2^\beta}_{\text{Sym}} e_3^\gamma d^3y$$

$$d\Sigma_\mu e_1^\mu \overset{=0}{=} d\Sigma_\mu e_2^\mu = d\Sigma_\mu e_3^\mu = 0$$

in space we have $\{e^\alpha_i, n\}$

$d\Sigma_\mu n^\mu \neq 0$ ∴ Can't be in the space of e^α_i

∴ $d\Sigma_\mu$ points in direction of



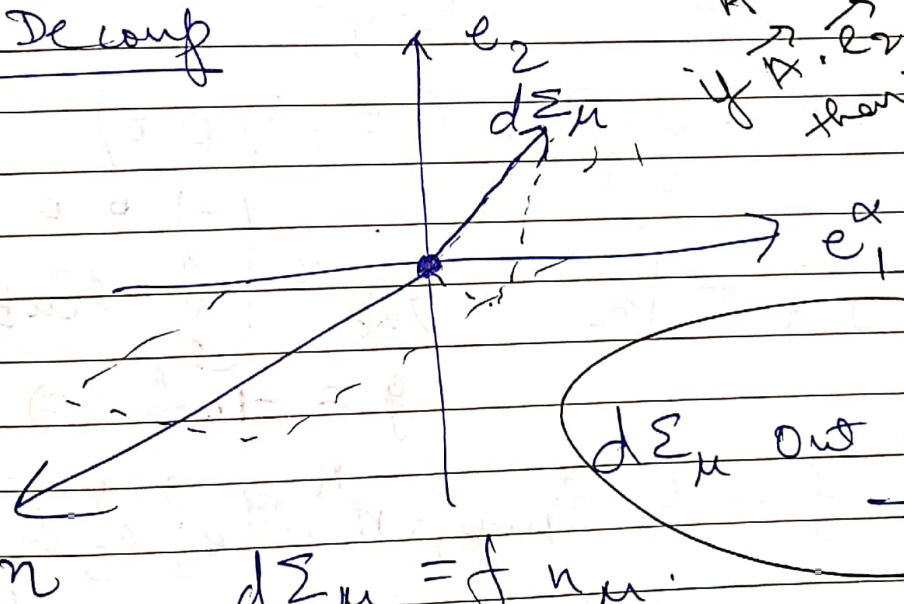
in similar vein $\vec{A}$ is in plane xy

∴ $\vec{A}$ is ⊥ z

$\vec{A} = a\vec{e_1} + b\vec{e_2}$

if $\vec{A}\cdot\vec{e_2} = 0$ then $\vec{A}$ is $\parallel \vec{e_1}$

2+1 Decomp



$d\Sigma_\mu$ out of e^a_i plane

$d\Sigma_\mu = f\, n_\mu$

~~But then why $d\Sigma_\mu$ in direction of n ?~~

Show $d\Sigma$

for non null case $d\Sigma_\mu = n_\mu \, d\Sigma$
$= n_\mu \sqrt{|h|}\, d^3y$.

(17) ~~Proof:~~ $d\Sigma_\mu = f n_\mu$.

$f n_\mu n^\mu = d\Sigma_\mu n^\mu \Rightarrow f\epsilon = d\Sigma_\mu n^\mu$

$f = \epsilon \, d\Sigma_\mu n^\mu$

$= \epsilon \, \epsilon_{\mu\alpha\beta\gamma} \, n^\mu e^\alpha_1 e^\beta_2 e^\gamma_3 \, d^3y$

Let spacetime metric be

Suppose $ds^2 = -dt^2 + h_{ab}\, dy^a dy^b$

$\Sigma = t = \text{const}$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & h_{ab} & \\ 0 & & & \end{pmatrix}$$

Doubt

$n^\alpha = (1, 0, 0, 0)$

$x^0 = t \qquad x^i = y^i$

$e^\alpha_1 = \dfrac{\partial x^\alpha}{\partial y^1} = (0, 1, 0, 0)$

$\sqrt{-g} = \sqrt{h} \qquad g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & h_{ab} & \\ 0 & & & \end{pmatrix}$

$g = -1 \times h \Rightarrow \sqrt{-g} = \sqrt{h}$

$f = \epsilon \, \epsilon_{\mu\alpha\beta\gamma} \, n^\mu e^\alpha_1 e^\beta_2 e^\gamma_3 \, d^3y$

$= (-1) \sqrt{-g} \, [\mu\alpha\beta\gamma] \, n^\mu e^\alpha_1 e^\beta_2 e^\gamma_3 \, d^3y$

$= \sqrt{h} \, [0123] \, d^3y$

$= \sqrt{h} \, d^3y$

(18) In general for S/T

$$d\Sigma_\mu = \epsilon\, n_\mu \sqrt{|h|}\, d^3y .$$

Proof:

This ϵ is nuisance

(19) $\boxed{\text{So why not add } \epsilon \text{ in Original form.}}$??

$$d\Sigma_\mu = \epsilon\, \epsilon_{\mu\alpha\beta\gamma}\, e^\alpha_1 e^\beta_2 e^\gamma_3\, d^3y .$$

(20) $$d\Sigma_\mu = \epsilon\, n_\mu \sqrt{|h|}\, d^3y$$

If Σ is Spacelike

& n_μ future pointing (Convention)

$\therefore\ \epsilon\, n_\mu = -n_\mu$

making $d\Sigma_\mu$ past Directed

for Timelike surface the $\epsilon = +1$

If n is outward

then $d\Sigma_\mu$ also

unlike $d\Sigma_\mu = \sqrt{h}\, n_\mu d^3y$

Doubt

(21) In Null Case

$$d\Sigma_\mu = \underbrace{\epsilon_{\mu\alpha\beta\gamma}\, e^\alpha_1 e^\beta_2 e^\gamma_3\, d^3y}_{\text{nothing acts funny in Null case}}$$

$e^\alpha_1 = k^\alpha \quad e^\alpha_A = \dfrac{\partial x^\alpha}{\partial \theta_A}$ ← Everything is well Defined

(22) Null Case

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha\, e_2^\beta\, e_3^\eta\, d^3y$$

Valid for any Coord. Syst & any T/S/N Cas

(23) $$d\Sigma_\mu = \underbrace{dS_{\mu\alpha}}_{} k^\alpha\, d\lambda$$

2D Surf element in Transverse Subspace

(24) Properties $dS_{\mu\alpha}$

(1) $dS_{\mu\alpha}$ is Antisymm as $dS_{\mu\alpha} \propto \epsilon_{\mu\alpha\beta\gamma}$

(2) $dS_{\mu\alpha}$ doesn't have components in e_2 & e_3.

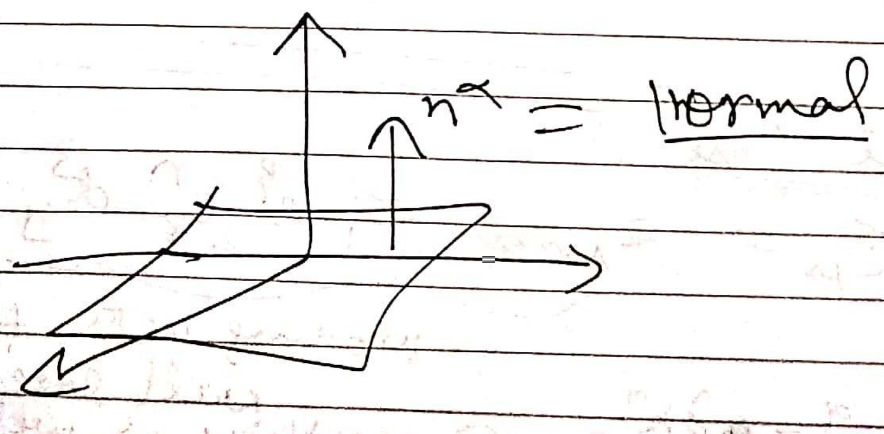
$\therefore$ It points in Direction of k & N.

$$dS_{\mu\alpha} = f\, k_{[\mu} N_{\alpha]}$$

(25) $$dS_{\mu\alpha} = 2\, k_{[\mu} N_{\alpha]} \sqrt{\sigma}\, d^2\theta$$

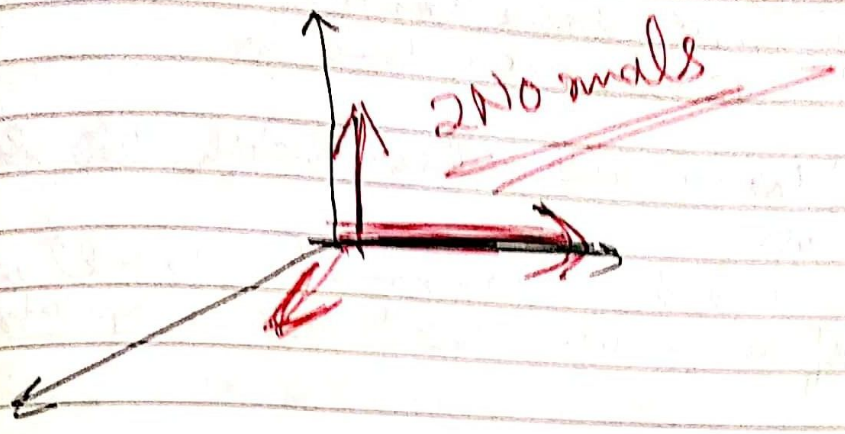
$$\sigma = \det(\sigma_{AB})$$

(26) If 2D Submanifold in 3D



Submanifold

1D in 3D



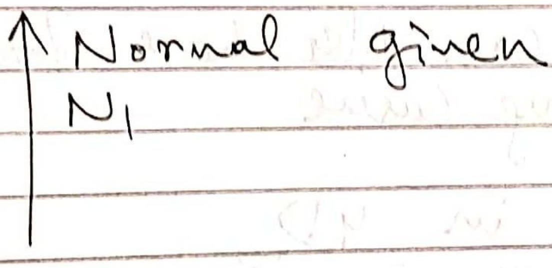
∴ in 2D

One Normal tells where the H.S is

in 1D

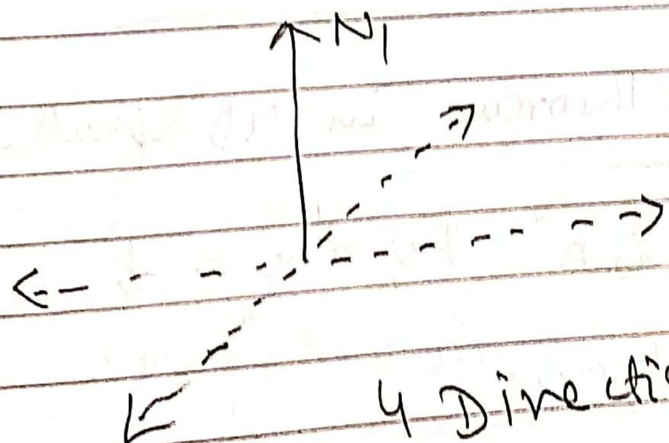
we need 2 Normals to tell the line

eg.



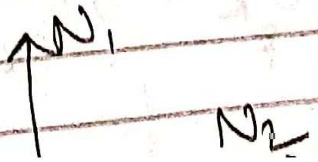
then

Line can be



4 Directions.

∴ One More Normal



&

then with some convention tell where the line is.

(27) Similar is the case with 2D subsurface in 4D

we need 2 Normals to tell which 2D space in 4D we are talking about

&

In Our Case

as $dS_{\mu\alpha}$ is not in direction of e_2, e_3

$\therefore$ 2 Normals we need to tell 2D transverse space is k, N.

(28) We can Also have Integeration measure along curve

1D in 4D

(29) $$d\Sigma_\mu = k_\mu \sqrt{\sigma}\, d^2\theta\, d\lambda$$

(30) Gauss Theorem in 4D spacetime

$$\int_{\mathcal{V}} \nabla_\alpha A^\alpha \sqrt{-g}\, d^4x = \oint_{\partial\mathcal{V}} A^\alpha d\Sigma_\alpha$$

Stokes Theorem:

$B^{\alpha\beta}$: A.S

$$\int_{\Sigma} \nabla_\beta B^{\alpha\beta} d\Sigma_\alpha = \frac{1}{2}\oint_{\partial\Sigma} B^{\alpha\beta} dS_{\alpha\beta}.$$

Proof of Gauss Theorem & Stokes in 4D & 3D

(31) How $dS_{\mu\alpha}$ is also the case of any 2D Transverse in 4D space?

$dS_{\mu\alpha}$ is n't limited only to Null Case &

$\to$ How?

(32) as in Non Null case H.S can be S/T

$\therefore |h|$

in Null case Transverse can be S/T

$\therefore |\sigma|$

(33) Conservation Statement

$$\nabla_\alpha J^\alpha = 0$$

let $J^\alpha = T^\alpha_\beta \xi^\beta$

$$\nabla_\alpha J^\alpha = \cancel{(\nabla_\alpha T^\alpha_\beta)} \xi^\beta + T^\alpha_\beta \nabla_\alpha \xi^\beta$$

if Energy Moment is conserv $\nabla_\alpha T^{\alpha\beta} = 0$

$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$ as $\mathcal{L}_\xi g_{ij} = 0$

$\beta \to$ A.S

J^α = Current Density

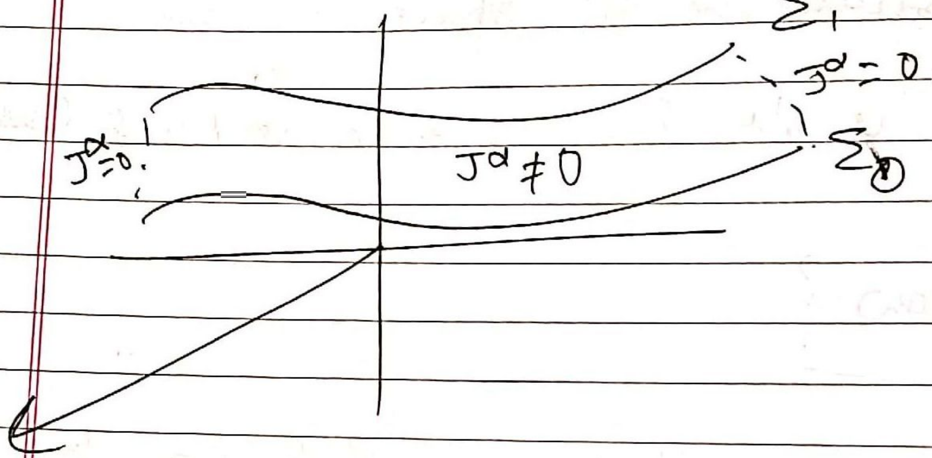
$\nabla_\alpha J^\alpha = 0$

$\downarrow$

Maxwell in Curved ST

$$\int_{\mathcal{V}} \nabla_\alpha J^\alpha \sqrt{-g}\, d^4x = 0 = \oint_{\partial\mathcal{V}} J^\alpha d\Sigma_\alpha$$

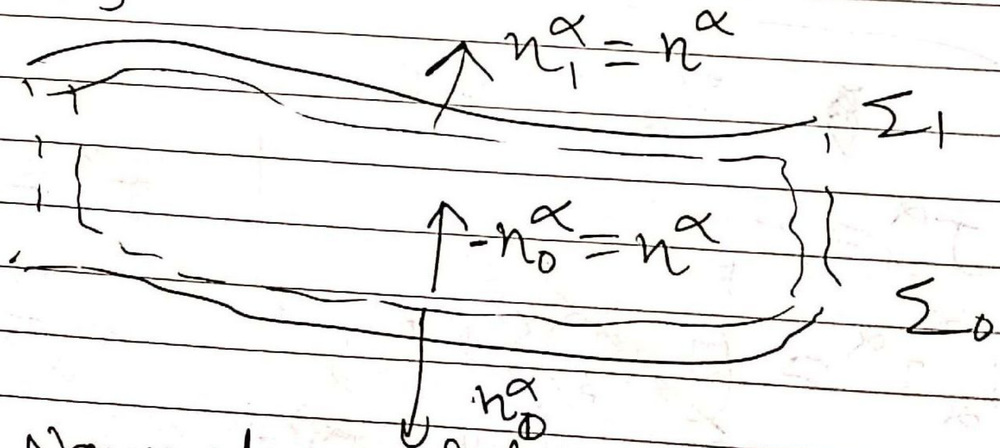
(34) we have Σ_1, Σ_2 2 spacelike H-S
we would connect them far away with timelike H.S.



We would say that $J^\alpha = 0$ at far away timelike H-S

$$\therefore \quad 0 = \int_{\Sigma_1} J^\alpha d\Sigma_\alpha + \int_{\Sigma_0} J^\alpha d\Sigma_\alpha$$

Taking smooth fn



Normal would point outward

But we want n_0^α to be flipped

$$0 = \int_{\Sigma_1} J^\alpha n_{1\alpha} \sqrt{h}\, d^3y + \int_{\Sigma_0} J^\alpha (-n_{0\alpha}) \sqrt{h}\, d^3y$$

$$\therefore \int_{\Sigma_0} J^\alpha n_{0\alpha} \sqrt{h}\, d^3y = \int_{\Sigma_1} J^\alpha n_{1\alpha} \sqrt{h}\, d^3y$$

$$\therefore Q = \int_{\Sigma} J^\alpha n_\alpha \sqrt{h}\, d^3y = \text{conserved}$$

$\Downarrow$

as Q is indep. of choice of Σ

(S) By $\nabla_\alpha T^{\alpha\beta} = 0$

There is no such conservation from this eqn