But we want n_0^α to be flipped

$$0 = \int_{\Sigma_1} J^\alpha n_{1\alpha} \sqrt{h}\, d^3y + \int_{\Sigma_0} J^\alpha (-n_{0\alpha}) \sqrt{h}\, d^3y$$

$$\therefore \int_{\Sigma_0} J^\alpha n_{0\alpha} \sqrt{h}\, d^3y = \int_{\Sigma_1} J^\alpha n_{1\alpha} \sqrt{h}\, d^3y$$

$$\therefore Q = \int_{\Sigma} J^\alpha n_\alpha \sqrt{h}\, d^3y = \text{conserved}$$

$\Downarrow$

as Q is indep. of choice of Σ

(5) By $\nabla_\alpha T^{\alpha\beta} = 0$

There is no such conservation from this Eq^n

23 Sept

Page 134

L-11

(1) Intrinsic & ~~Extrinsic~~ Geometry:

In Spacetime we have metric $g^{\alpha\beta}$, we have Connection Γ

The curvature comes from then $R^{\alpha}_{\beta\gamma\delta}$

(2) In H.S y^a

we have h_{ab}, we have Connection Γ^a_{bc}

from this we get Intrinsic Curv. of H.S R^a_{bcd}

(3) $\boxed{\begin{array}{l}\text{Relation B/W Both } \Gamma \\ \text{Relation B/W Spacetime } R \text{ \& Intrinsic } R\end{array}}$

Spacetime (Bulk) Curvature = Intrinsic Curvature + Extrinsic Curvature

evaluated on H.S

→ Gauss Codazzi

(4) R^a_{bcd} is the geometry of the H.S

(5) In Book everything is done in covariant fo[rm]
i.e.
We don't want to know relationship B/W x^α & y^a
Though there will be some relationship
But I don't want to know what it is.

(6) In the lecture
I will assume ~~some~~ relationship by taking suitable coordinate system.

(7) Assuming H.S. is spacelike.

In Null Case Relationship B/w Extrinsic & Intrinsic curvature is subtle.

(8) To work in the neighborhood of the H.S. choose Gaussian Normal coordinates. (X^α)

∴ The spacetime coordinates system would be fixed.

But ~~take~~ there is freedom to choose intrinsic coordinates.
i.e. Everything is covariant w.r.t. $y^a \to y^{a'}$.

(9) Intrinsic Covariant Derivative

Defined on H.S. with intrinsic connection

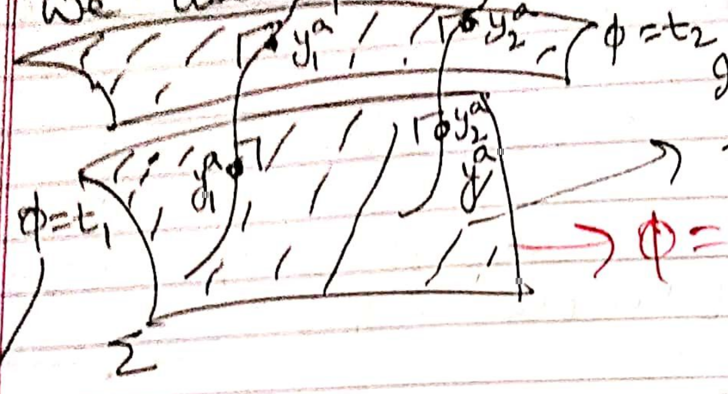
$|a$ ≡ Intrinsic Covariant Derivative.

when ~~$D_a h_{bc} = 0$~~ $h_{bc|a} = 0$ then we say connection is compatible with Intrinsic metric on H.S.

$$h_{bc|a} = 0 \equiv \nabla_\alpha g_{\beta\gamma} = 0$$

Gaussian Normal Coord. (x^α)

(10) We use GNC in the Neighborhood of H.S



Intrinsic coordinates have freedom

$\Rightarrow \phi = t_1 = 0$

$\Rightarrow$ H.S. ~~where~~ of proper time fixed.

Imp $\Rightarrow$ Assuming y^a Remains same along the curve just proper time changes.

$\Rightarrow$ Assuming all curves are H.S. O.

$\Rightarrow$ Not compulsory for curves to be Geodesic

Parameters

GNC : x^α : $x^0 = t$ = proper time
$x^a = y^a$

(11) GNC turns out to be Really Beneficial locally around H.S. when I am interested all on H.S.

(12) Metric Spacetime in GNC

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = dt^2 - g_{ab}(t, x^a)\, dx^a dx^b$$

Metric from Analysis

⇒ Moving just along curves $dx^a = 0$ & only proper time is changing
which is also evident from metric

$$ds^2 = dt^2 - g_{ab}\,dx^a dx^b$$

$$dx^a = 0$$

$$\underline{ds^2 = dt^2} \text{ along the } \underline{\text{curves}}$$

⇒ But By above argument we can still have $dt\,dx^a$ terms in the metric

$$g_{\alpha\beta} \overset{*}{=} h_{\alpha\beta} + U_\alpha U_\beta$$

Comoving frame

↳ originated in Minkowski & in flat $U_\alpha \perp h_{\alpha\beta}$
But as they are
tensors $g_{\alpha\beta} = h_{\alpha\beta} + U_\alpha U_\beta$

though U_α is $\perp^r$ or not to H-S.
This is valid

Similarly
$$g_{\alpha\beta} \overset{*}{=} h_{\alpha\beta} + \epsilon n_\alpha n_\beta$$

↳ in Mink spacetime as $U_\alpha \perp h_{\alpha\beta}$ & $U_\alpha = n_\alpha$

$$\therefore\ \underline{g_{\alpha\beta} = h_{\alpha\beta} + \epsilon n_\alpha n_\beta} \quad \text{in General}$$

But Now in this situation
U_α is $\perp^r$ to H-S in GNC coord.

$$\therefore\ g_{\alpha\beta} = h_{\alpha\beta} + U_\alpha U_\beta \quad \text{where } U_\alpha \perp h_{\alpha\beta}$$

$$\therefore\ g_{\alpha\beta} = \begin{pmatrix} & \\ & \boxed{h_{\alpha\beta}} \end{pmatrix} = U_\alpha U_\beta + h_{\alpha\beta}$$

as $U_\alpha \perp h_{\alpha\beta}$

$U_\alpha = (a, 0, 0, 0)$

$\Rightarrow$ Only dt^2 coeff in ds^2

all others $dt\, dx^a = 0$

(13) if $t = 0$ on the H.S. $\qquad ds^2 = dt^2 - g_{ab}\, dx^a dx^b$

$g_{ab}(X^0 = 0, X^a) = h_{ab} \qquad g_{ab}(t, x^a)$

Spacetime metric = Induced metric

if $t = t_2 \neq 0$ other H.S. is considered

$g_{ab}(X^0 = t_2, X^a)$

But on our H.S. $t = 0$

$\boxed{g_{ab} = h_{ab}}$

Because $\quad ds^2\Big|_{\substack{H.S.\\ t=0}} = g_{\alpha\beta}(t, x^a)\, dx^\alpha dx^\beta$

$= g_{ab}(t=0, x^a)\, dx^a dx^b$

If h_{ab} depends on time? $h_{ab}(t, y^a)$? But By Def: $\quad ds^2\Big|_{\substack{H.S\\ t=0}} = h_{ab}\, dy^a dy^b$

But $y^a = x^a$

$\therefore\ g_{ab}(t=0, x^a) = h_{ab}$

Does this coordinate system work if there is singularity?

(14) GNC coordinates help us to get metric on H.S. which is easy.

(15) We can also take GNC in 2D H.S.

Here 4D ⟶ 3D
But also 3D ⟶ 2D

(16) Using GNC calculating Γ, $R^a{}_{bcd}$

(17) $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ab} \end{pmatrix}$

$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -g^{ab} \end{pmatrix}$ { Proof }

(18) $^4\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\mu}}{2}\left(-\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta}\right)$

$\alpha = 0, 1, 2, 3$

calculating it: $^4\Gamma^0_{\alpha 0} = 0$ $\quad ^4\Gamma^\alpha_{00} = 0$

$^4\Gamma^0_{ab} = \frac{\partial_0 g_{ab}}{2}$

$^4\Gamma^a_{tb} = \frac{g^{am}}{2}\partial_t g_{mb}$

$^4\Gamma^a_{bc} = \frac{g^{am}}{2}\left(-\partial_m g_{bc} + \partial_b g_{cm} + \partial_c g_{mb}\right)$

Valid anywhere around the H.S.

(19) $t=0$ making on the H-S: $\phi = 0$

$${}^4\Gamma^0_{ab} = K_{ab} = \frac{\partial_0 g_{ab}}{2}\Big|_{t=0}$$

Extrinsic Curvature

$${}^4\Gamma^a_{0b} = \frac{1}{2} g^{am}(t=0)\, \partial_0 g_{mb}\Big|_{t=0}$$

$$= \frac{h^{am}}{2}\, \partial_0 g_{mb}\Big|_{t=0}$$

$$= h^{am} K_{mb}$$

$$= K^a_b$$

Is $h_{ab}(t, y^a)$ dependent on time?

(13) See (13)

$${}^4\Gamma^a_{bc} = \frac{h^{am}}{2}\left(-\partial_m g_{bc}\Big|_{t=0} + \partial_b g_{cm}\Big|_{t=0} + \partial_c g_{mb}\Big|_{t=0}\right)$$

h_{ab} is scalar, $x^a \to x^{a'}$ cuz $y^a \to y^{a'}$

$${}^4\Gamma^a_{bc} = {}^3\Gamma^a_{bc}$$

(20) ${}^4\Gamma$ are not tensors in spacetime Coord

Proof? ${}^4\Gamma$ are not tensors in Intrinsic Coord

But ${}^4\Gamma^t_{ab}$ are 3-Tensors in Intrinsic Coord $y^a \to y^{a'}$

Proof? $\therefore$ as ${}^4\Gamma^0_{ab} = K_{ab}$

h_{ab} Scalar w.r.t $x^a \to x^{a'}$; K_{ab} is 3 Tensor under $y^a \to y^{a'}$

(21) By using the Tangent vectors we can promote
3-Tensor ⟶ 4 Tensors
&

we can also pullback from 4 Tensor ⟶ 3 Tensor

∴ K_{ab} can also be defined as 4 Tensor
∴ There is one-one Correspondence b/w 4-T → 3-T.

(22) Here in this GNC specified coord. system
we can't prove
K_{ab} as Tensors under $y^a \to y^{a'}$

See 135 (19) (43)

From Covariant, we can prove this.

(23) As $${}^4\Gamma^0_{ab} = \frac{\partial_0 g_{ab}}{2}\Big|_{t=0} = K_{ab}$$ symm

Kab is Symm from Covariant approach?
$(\nabla_\beta e^\alpha_a) e^\beta_b = e^\beta_b \nabla_\beta e^\alpha_a$
$e^\beta_b(\nabla_\beta e^\alpha_a) = e^\beta_a \nabla_\beta e^\alpha_b$

Basis vectors Lie transport

∴ ${}^4\Gamma^0_{ab}$, K_{ab} is symm

(24) On Σ $\phi = 0$ H.S.

Why Not Tensorial on Σ?

~~as K_{atb} is not tensor on H.S.~~

$${}^4R_{tatb} = -\frac{1}{2}\partial_t^2 g_{ab}\Big|_{t=0} + K_{am}K^m{}_b$$

from ΓΓ terms

(3) Compare with (3.3.4)

~~as ${}^4R_{tabc}$ is not tensor on H.S.~~

$${}^4R_{tabc} = K_{ab|c} - K_{ac|b} \quad (1)$$

See 135 (19) (43)

∴

How do I know $K_{ab|c}$ is Tensorial on Σ?

Spatial Bulk Comp. of Curvature

On R.H.S. Intrinsic Curv + Extrinsic Curv

(3) — $^{4}R_{abcd} = {}^{3}R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}$

Compare with (3.29)

Intrinsic Riemann Tensor

(1) & (2) are Tensorical Eqn on Σ

(1) & (2) are called Gauss Codazzi Eqn

(25) (1) & (2) has one time Derivative of $g_{ab}\big|_{t}$

K_{ab} ↔ spatial

$$K_{ab} = \frac{\partial_0 g_{ab}}{2}\Big|_{t=0}$$

But (3) has 2 time Derivative of $g_{ab}\big|_{t=0}$

→ Useful in Initial Value Problem

(26) Extrinsic Curvature tells Bending of submanifold embedded in 4D.

(27) Till Now All these (1) & (2) & (3) Eqns are valid only in GNC

→ (1) & (2) are Covariant under $y^a \to y'^a$ But not under $x^\alpha \to x'^\alpha$

we will see that (1) (2) (3) are covariant $x^\alpha \to x'^\alpha$

classmate
Date
Page 142

(28) Einstein Tensor on Σ $\phi = 0$

(4) — $^{4}G_{tt} = \frac{1}{2}\left({}^{3}R - k^{ab}k_{ab} + k^{2}\right)$

(5) ← $^{4}G_{ta} = k^{ab}_{a|b} - k_{|a}$

$k = Tr\, k^{a}_{b} = k^{a}_{a} = h^{ab}k_{ab}$

(6) — $^{4}G_{ab} = {}^{3}G_{ab} + \frac{\partial^{2}_{t} g_{ab}(t=0)}{2}$

$- \frac{h_{ab}}{2}\left(h^{cd}\partial^{2}_{t} g_{cd}\Big|(t=0)\right) - 2k_{ac}k^{c}_{b}$

$+ \frac{3h_{ab}}{2}\left(k^{cd}k_{cd}\right) + k k_{ab} - \frac{h_{ab}k^{a}}{2}$

Prove all this ?

(29) All $g_{\mu\nu}$ are valid in GNC

But Intrinsic Coordinates were choosen Arbitrarily.

Only ~~Ext. Co~~ Spacetime coord. were fixed.

(30) To turn all this 4D. covariant exp. $x^{\alpha} - x^{\alpha'}$

Reintroduce Basis vectors on H.S.

$\left(n^{\alpha}, e^{\alpha}_{a}\right)$

in GNC Def: $n_{\alpha} \propto \partial_{\alpha}\phi$

→ $\phi = t = 0$ ⇒ $n_{\alpha} \stackrel{*}{=} (1,0,0,0)$

$n^{\alpha} \stackrel{*}{=} (1,0,0,0)$

Normalized?,

tive
e in
action
$+\delta f$,
then
m is
$)-f(x)$
$)\epsilon + \frac{\epsilon f^{2}}{2}$
s defined
$=0$
ration
s

in GNC

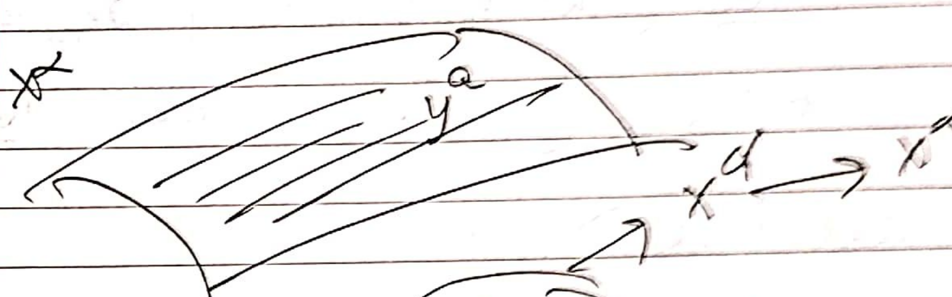
$$e^{\alpha}_{1} \overset{*}{=} (0,1,0,0)$$

$$e^{\alpha}_{2} \overset{*}{=} (0,0,1,0)$$

$$e^{\alpha}_{3} \overset{*}{=} (0,0,0,1)$$

$$K_{ab} \overset{*}{=} \left.\frac{\partial_t g_{ab}}{2}\right|_{t=0}$$

(31) Two sets of coordinate freedom



We can have x^α Arb. y^a fixed

x^α fix y^a Arb. $y^a \to y^{a'}$

x^α & y^a fixed

? What are the examples ?

(32) Tensors w.r.t. spacetime coord are scalar w.r.t $y^a \to y^{a'}$

Doubt

Tensors w.r.t $y^a \to y^{a'}$ are scalars w.r.t. $x^\alpha \longrightarrow x^{\alpha'}$ — e.g. h^{ab}, K_{ab}

$$h^{\alpha\beta} = h^{ab} e^{\alpha}_{a} e^{\beta}_{b}$$

(33) Def $K_{ab} \equiv \nabla_{(\beta} n_{\alpha)} e^{\alpha}_{a} e^{\beta}_{b}$ → Projection Oper

Scalar under $x^\alpha \to x^{\alpha'}$ → Projection on 3D

Evaluate at GNC & see if matches

classmate

Date
Page

144

(34) $\nabla_\beta n_\alpha = \partial_\beta n_\alpha - {}^4\Gamma^{\gamma}_{\alpha\beta} n_\gamma$

In GNC

$\phi = t =$ const.

$n_\alpha = \partial_\alpha \phi = (1,0,0,0)$

$\therefore \partial_\beta n_\alpha = 0$

$\nabla_\beta n_\alpha \overset{*}{=} -{}^4\Gamma^0_{\alpha\beta}$

$\nabla_{(\beta} n_{\alpha)} = \dfrac{\nabla_\beta n_\alpha + \nabla_\alpha n_\beta}{2}$

$\nabla_{(\beta} n_{\alpha)} \overset{*}{=} -{}^4\Gamma^0_{\alpha\beta}$

how this is scalar

$\nabla_{(\beta} n_{\alpha)} e^\alpha_a e^\beta_b \overset{*}{=} -{}^4\Gamma^0_{\alpha\beta}\delta^\alpha_a \delta^\beta_b - {}^4\Gamma^0_{ab} \overset{*}{=} K_{ab}$

Scalar in spacetime

Equality of scalar in GNC

$\Rightarrow$ Equality in all Coord

Cov.

$\therefore$ Def. of K_{ab} is valid

(35) Can we go from specific GNC $\rightarrow$ Covariant Def ? for K_{ab}

(36) Now Converting Rieman Tensor Eqn (Gauss Codazzi) & Contracted Einsten Tensor Eqn's to Covariant form →

(1) & (6) cannot be written in covariant form. as they contain 2nd time derivative of metric
& we don't have any geometrical structure analogous to $\partial_t^2 g_{ab}\big|_{t=0}$

$\Rightarrow K_{ab} \equiv \nabla_{(\beta} n_{\alpha)} e^{\alpha}_{a} e^{\beta}_{b}$

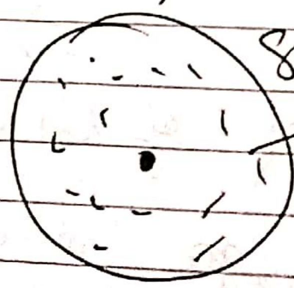
→ Gradient of normal vector along H-S

→ Tells us how normal vector varies along H-S

But to tell how normal vector varies away from surface, we don't know
as
Normal is Only defined on H-S. Not away from H-S.

∴ Derivatives ∇_β along those direction away from ~~norma~~ H.S. Not Defined.

Only on H-S.



Sphere → all Direction

$\partial_t^2 g_{ab}\big|_{t=0}$

(why) To tell anything About this I have to know what is happening in the neighborhood of HoS But from the given Geometrical structure I don't have enuf information.

37) Remaining Covariant Expression

${}^4R_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha_a e^\beta_b e^\gamma_c$ is equivalent to ${}^4R_{tabc}$ in GNC

as $\left\{ K_{ab|c} - K_{ac|b} \right.$ is spacetime scalar

$\therefore$ LHS ${}^4R_{tabc}$ is scalar $x^\alpha \to x^{\alpha'}$ $\therefore$ valid in any frame

$$\therefore\ {}^4R_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha_a e^\beta_b e^\gamma_c = K_{ab|c} - K_{ac|b}$$

Similarly

$${}^4R_{\alpha\beta\gamma\delta}\, e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}$$

Similarly

Contracted $\left\{ \begin{aligned} {}^4G_{\mu\nu}\, n^\mu n^\nu &= \tfrac{1}{2}\left(R - K^{ab}K_{ab} + K^2\right) \\ {}^4G_{\mu\alpha}\, n^\mu e^\alpha_a &= D_b K^b{}_a - D_a K \end{aligned} \right.$

Gauss Codazzi

Momentum Constraint?

Initial Value problem L-12

(1) Mechanics $m\ddot{x} = F$

$\therefore$ 2 Initial values to get Unique soln $\leftarrow$ 2nd order DE

Unique soln only if Initial value is provided

eg: $x(0)$; $\dot{x}(0)$ same

Only by eqn we can get many soln But we get Unique soln if Initial values are given

(2) field theory

Wave eqn

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = \rho \rightarrow \text{source}$$

$$\Rightarrow \nabla^2 \phi - \rho = \frac{\partial^2 \phi}{\partial t^2}$$

In Mechanics we have finite DoF but in field theory we have ∞ DoF as ϕ has value at every spatial position.

For Unique soln to field eqn

Initial Value: $\phi(t=0, \vec{x})$; $\partial_t \phi(t=0, \vec{x})$ same

(3) By theorems of Uniqueness & Existence of PDE, we have well posed problem if $\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \rho$ & 2 Initial values

(a) We are trying to get the equivalent of all this in GR.

One Problem in field theory eqn was that eqn was not Covariant.

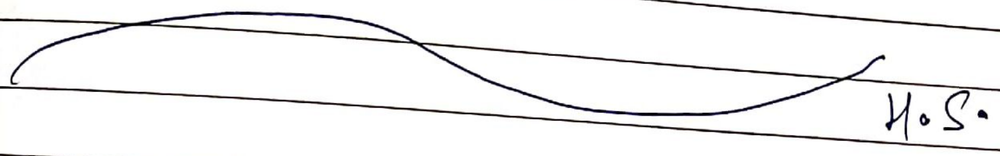
$\therefore$ Make Covariant scalar field eqn

in GR

(1) $-g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = \rho \Leftrightarrow \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = \rho$

And for Initial value

we have to take aribitrary ~~timelike~~ spacelike H.S. & initial data has to be provided there. So that eqn (1) becomes well posed.



ϕ : Scalar field

Φ : H.S.

Initial values : $\Phi(y^a) = \phi\big|_\Sigma \Rightarrow$ Covariant version of $\phi(t=0,\vec{x})$

Normal component of Derivative of scalar $\Big\} \Rightarrow n^\alpha\partial_\alpha\Phi(y^a)\Big|$

Uniqueness & Existence Theorem tells that (1) Eqn along with these Initial values provide well posed problem.

Nov-22

L-13

1. For finite degree of freedom can be
Generalized coordinate $q(t)$: ① linear position variable
② Angular position variable
⋮

Generalized velocity $\dot{q}(t)$

Lagrangian function : $\mathcal{L}(q, \dot{q})$

Action functional $S[q] = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q})\, dt$

② Def: Functional : $f \xrightarrow[F]{} \mathbb{R}$

Here $q \xrightarrow[S]{} \mathbb{R}$

q ↑ function, $\mathbb{R}$ ↑ Number.

③ EOM : Hamilton's principle $\delta S = 0$
Action would be stationary in the neighborhood of the true path.

Or

True path would be extremum of Action.

Initial condition $\delta q(t_1) = \delta q(t_2) = 0$

If this is not then we won't have well defined variational principle

(4) Euler Lagrange Equation

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

(5) Scalar field theory in Curved spacetime

$\left\{ \begin{array}{l} \phi(x^i) \\ \partial_\alpha \phi(x^i) \end{array} \right. \longrightarrow \infty$ no. of dof.

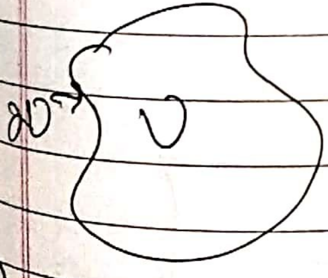
Lagrangian Density = Lagrangian per Unit Vol.

$$\mathcal{L}(\phi, \partial_\alpha \phi) = \text{Scalar fn}.$$

(6) Action fnal $\quad S[\phi] = \int_V d^4x \, \mathcal{L} \sqrt{-g}$

V = fixed finite 4D region

∂V : Boundary closed 3 Surface



Why not vary this?

(7) $\delta\phi = 0$ on ∂V

$$\delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\underbrace{\partial_\alpha \delta\phi}_{\nabla_\alpha \delta\phi} \right) \underbrace{\sqrt{-g}\, d^4x}$$

Keeping the Boundary fixed [illegible] laws.

$$= \oint_{\partial V} \cancel{\left(\frac{\partial \alpha}{\partial \phi_{,\alpha}} \delta\phi\right) d\Sigma_\alpha} + \int_V \left(\frac{\partial \alpha}{\partial \phi} - \nabla_\alpha \frac{\partial \alpha}{\partial \phi_{,\alpha}}\right) \delta\phi \sqrt{-g}\, d$$

$$\therefore \quad \frac{\partial \alpha}{\partial \phi} - \nabla_\alpha \frac{\partial \alpha}{\partial \phi_{,\alpha}} = 0$$

(8) **Example:** $\alpha = +\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi)$

$\boxed{\text{Why + sign ?}}$

Going to LIF.

$$\alpha = \frac{\eta^{\alpha\beta}}{2} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi)$$

$$\eta^{00} = 1$$

$$\therefore \alpha = \frac{\partial_t^2 \phi}{2} - V(\phi) - - - -$$

$\partial_t \phi$ has to come with +ve sign $\therefore$ + sign there.

If our convention ~~$\eta_{\alpha\alpha}$~~ $- + + +$
$\eta^{00} = -1$ $\therefore$ There has to be $-$ve sign

(9) $\nabla_\alpha \left(\frac{\partial L}{\partial \phi_{,\alpha}} \right) = + g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi \equiv - \Box \phi$

$\Box \equiv \nabla_\alpha \nabla^\alpha$

$\boxed{\text{As } \phi \text{ is Ind of } \phi_{,\alpha}}$

$\therefore \frac{\partial \mathcal{L}}{\partial \phi} = - \frac{dV}{d\phi}$

field Eqn $\therefore \Box \phi = \frac{dV}{d\phi}$

free Massless field $V = 0$

free Massive field $V = \frac{m^2}{2} \phi^2$

we would get linear Eqn

As it is linear we call it free

This would yield Klein Gordon Eqn in Curved space

Interacting field $V = \frac{m^2}{2} \phi^2 + \lambda \phi^4$

Self Interacting

gives Non Linear Eqn $\therefore$ Interacting

(10) for Vector field

classmate
Date
Page

(1) $S = S_g[g] + S_m[\phi, g]$

∂V is almost nowhere null.

$S_g = \frac{1}{16\pi}\int_V R\sqrt{-g}\, d^4x + \frac{1}{8\pi}\oint_{\partial V} \epsilon K |h|^{1/2} d^3y$

$S_m[\varphi, g] = \int_V \mathcal{L}(\varphi, \partial_\alpha \varphi)\sqrt{-g}\, d^4x$

In field theory
there was dependence of $\phi, \phi_{,\alpha}$ not higher order
here it should be dependent on $g, \partial_\alpha g$.
But $\Gamma = (g, \partial_\alpha g)$
is not scalar But $\mathcal{L}$ has to be scalar
$\therefore$ to construct a scalar $\mathcal{L}$ put R in S.

Scanned with CamScanner

Functional Derivative

① Functional derivative or variational derivative relates a change in functional to a change in a function on which functional depends.

② Let the functional :

$$J[f] = \int_a^b L(x, f(x), f'(x))\, dx$$

If the f is varied adding to it a function δf and the resulting integrand $L(x, f+\delta f, f'+\delta f')$ is expanded in powers of δf, then the coefficient of δf in first order term is called functional derivative δJ.

$$\delta J = \int_a^b \left(\frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'} \frac{d}{dx}(\delta f) \right) dx \qquad f(x+\epsilon) - f(x) = f'(x)\epsilon + \frac{\epsilon^2}{2} f''$$

Definition :

③ Functional derivative of $F[\rho] \equiv \frac{\delta F}{\delta \rho}$ is defined as :

$$\int \frac{\delta F}{\delta \rho} \phi(x)\, dx = \lim_{\epsilon \to 0} \frac{F[\rho + \epsilon\phi] - F[\rho]}{\epsilon}$$

$$= \left. \frac{d}{d\epsilon} F[\rho + \epsilon \phi] \right|_{\epsilon = 0}$$

where ϕ is arbitrary function. & $\epsilon\phi$ is variation of ρ

④ The differential of the functional $F[\rho]$ is (b)

$$\delta F[\rho ; \phi] = \int \frac{\delta F}{\delta \rho} \phi\, dx$$

Let $\phi = \delta \rho$ $\therefore$ ϕ be change ρ

⑤ Similarity to Total Differentiation.

Let F be function $F(\rho_1, \rho_2 \ldots \rho_n)$

$$dF = \sum \frac{\partial F}{\partial \rho_i} d\rho_i$$

where $\rho_1, \rho_2 \ldots \rho_n$ are independent variables.

But

$$\delta F[\rho] = \int \left(\frac{\delta F}{\delta \rho}\right) (\underset{\phi}{\delta \rho})\, dx$$

$$\frac{\delta F}{\delta \rho} \equiv \frac{\partial F}{\partial \rho_i}$$

Eg.

⑥ Let the functional be

$$F[\rho] = \int f(\vec{r}, \rho(\vec{r}), \nabla \rho(\vec{r}))\, d\vec{r}$$

$$\delta F[\rho; \phi_1, \phi_2] = \int \frac{\delta F}{\delta \rho} \phi(\vec{r})\, d\vec{r} = \left[\frac{d}{d\epsilon} F\right]_{\epsilon = 0}$$

$$= \left[\frac{d}{d\epsilon} \int f(r, \rho(r) + \epsilon\phi, \nabla\rho + \epsilon\nabla\phi)\, d\vec{r}\right]_{\epsilon=0}$$

$$= \int \lim_{\epsilon \to 0} \frac{f(r, \rho + \epsilon\phi, \nabla\rho + \epsilon\nabla\phi) - f(r, \rho, \nabla\rho)}{\epsilon}\, dr$$

$$= \int \lim_{\epsilon \to 0} \frac{f(r, \rho, \nabla\rho) + \frac{\partial f}{\partial \rho}\phi\epsilon + \frac{\partial f}{\partial(\nabla\rho)}\nabla\phi\epsilon - f}{\epsilon}\, dr$$

Taylor Exp^n

① $f(x)$ around x_0

$$f(x) = f(x_0) + f'(x)\Big|_{x_0}(x - x_0) + f''(x)\Big|_{x_0}\frac{(x - x_0)^2}{2!} \ldots$$

Around x_0

$\uparrow$

(2) $f(x_0+\epsilon) = f(x_0) + f'(x_0)\,\epsilon + f''(x_0)\dfrac{\epsilon^2}{2!} \ldots$

$$\vec{\nabla}\phi \qquad = \int\left(\frac{\partial f}{\partial \rho}\phi + \frac{\partial f}{\partial(\nabla\rho)}\nabla\phi\right) d\vec{r}$$

$\parallel$

Derivative of Scalar w.r.t. vector

$$= \int\left[\frac{\partial f}{\partial \rho}\phi + \vec{\nabla}\cdot\left(\frac{\partial f}{\partial(\nabla\rho)}\phi\right) - \left(\nabla\cdot\frac{\partial f}{\partial\nabla\rho}\right)\phi\right] d\vec{r}$$

$\uparrow$

Let $\dfrac{\partial f}{\partial(\vec{\nabla}\rho)} = \vec{A}$

$$\int\left(\nabla\cdot\frac{\partial f}{\partial(\nabla\rho)}\phi\right) dr = \int(\nabla\cdot\vec{A})\,\phi\, d^3x = \int_{\partial V}\vec{A}\cdot\vec{n}\,\phi\, d^2x$$

Let on Surface $\phi = 0$

i.e. $\delta\rho$ on Surface $= 0$

$$\therefore \qquad = \int\left(\frac{\partial f}{\partial \rho}\phi + \vec{\nabla}\cdot\frac{\partial f}{\partial(\nabla\rho)}\phi\right) d^3x$$

& $$\frac{\delta F}{\delta \rho} = \frac{\partial f}{\partial \rho} - \nabla\cdot\frac{\partial f}{\partial\nabla\rho}$$

(7) Fundamental Lemma of Calc. of Variation

$\int_a^b f(x)\,h(x)\,dx = 0$ if f is cont. fn in (a,b)

for smooth $h(x)$ in (a,b) then $f(x) = 0$

L-15

① $p = \frac{\partial L}{\partial \dot{q}} = f(q, \dot{q})$ Assuming $\dot{q} = g(q, p)$ can be extracted

$H = p\dot{q} - L = H(p, q)$

② Vary the Lagrangian. $L = p\dot{q} - H(p, q)$

$$\delta_{(p,q)} S = \int_{t_1}^{t_2} \delta(p\dot{q} - H)\, dt$$

$$\delta q(t_1, t_2) = 0 \Rightarrow \left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} \end{aligned}\right\} \text{Hamilton's Equation}$$

③ But here we are interested in getting Hamiltonian & not Hamilton's Eqn
∴ Construct Hamiltonian for field theory & GR.

④ Field theory in flat spacetime

$$\mathcal{L}(\phi, \partial_\alpha \phi) \longleftrightarrow \pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

selected the frame ∴ Lorentz invariance is gone away.

At level of Hamiltonian Lorentz Invariance Over

→ $\mathcal{H}(\phi, \pi)$ functional of ϕ, π

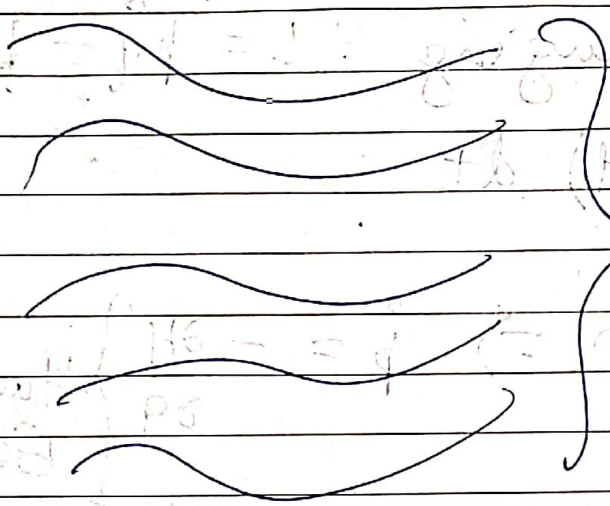
⑤ $\mathcal{H} = \pi^0 \partial_0 \phi - \mathcal{L} = T_{00}$

$H = \int \mathcal{H}\, d^3x$ = Total Energy of field

To express action in terms of the Hamiltonian it is necessary to foliate V with a family of spacelike H.S..

(5) ~~field theory in Curved Spacetime~~

foliate spacetime with arbitrary spacelike Hypersurfaces.



Any shape
Only condition
: It doesn't Intersect & they are spacelike.

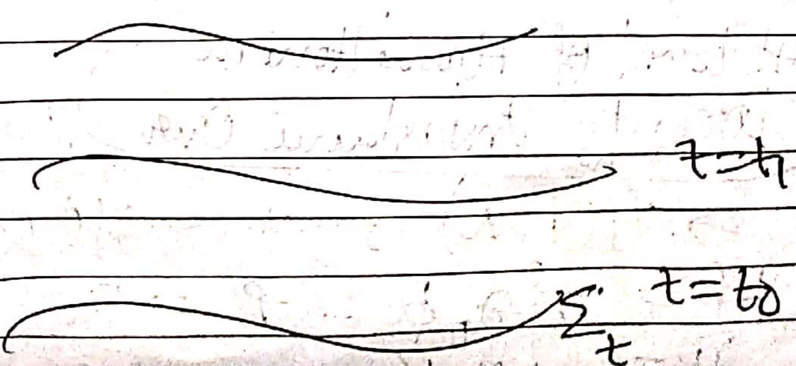
(6) Analog of time derivative (how field are changing) ~~along the H.S~~ is the normal derivative of field

$$\partial_0 \phi \longrightarrow n^\alpha \partial_\alpha \phi$$

(7) Define time function $\longrightarrow$ scalar field.

$t(x^\alpha)$, s.t. $t =$ const on each Σ_t

$$\boxed{x^0 \neq t}$$



t is a good candidate to call time because it keeps on increasing on the spaclike surfacds

(8) I have the freedom to use $t = x^{0'}$ in $(t, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}})$ as new coordinats instead of (x^0, x^1, x^2, x^3) due to general covariance.
But in general $t = x^0$ Still t is a good function to be called time as ↑ in surfac

(9) In Our coord. System

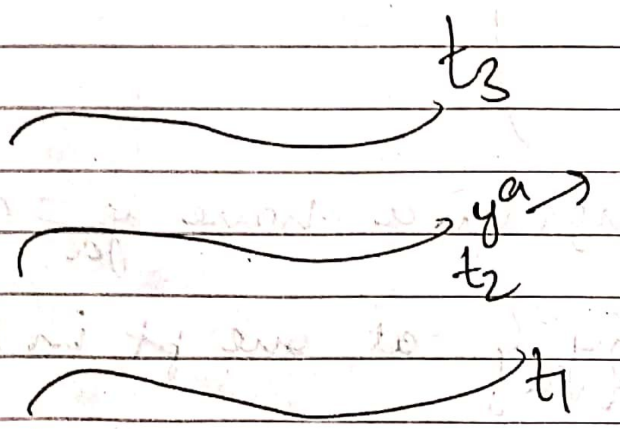
$$t(x^\alpha) = \text{const.} \quad \text{on each } \Sigma$$

$\therefore$ t is just a Scalar

(10) $n_\alpha \propto \partial_\alpha t$

has to be Normalized → $\boxed{n_\alpha n^\alpha = 1}$

(11)



Intrinsic coord. on Σ & coord. could be diff. on diff. H.S.

like on $t = t_1$ we use Cartesian (x, y, z)

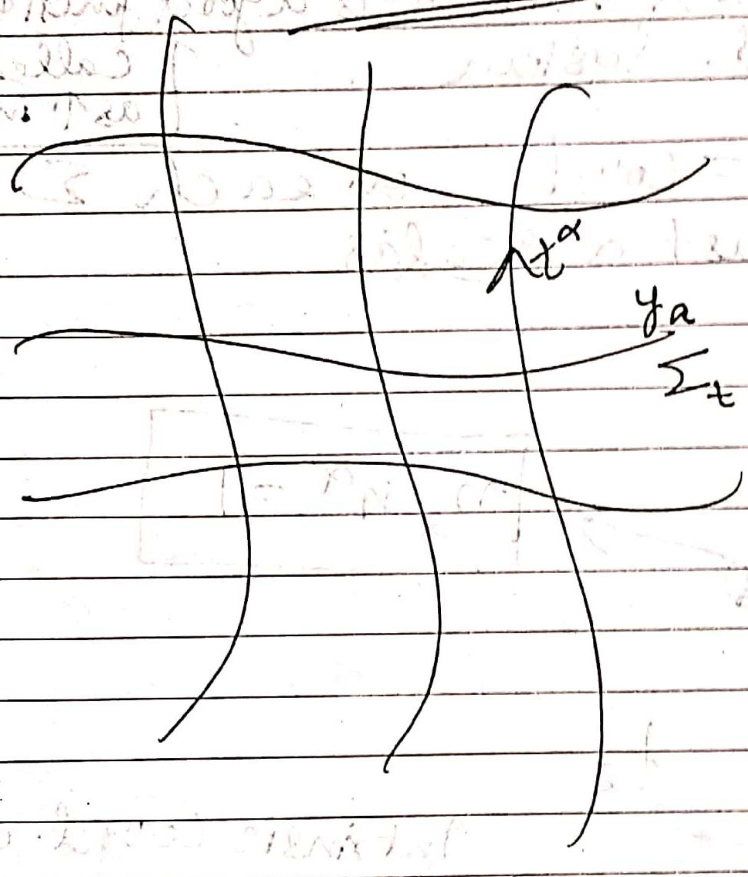
& on $t = t_2$ we use Sph. (r, θ, ϕ)

But all this will Cause Difficulty for noReason

PAGE NO.:
DATE: / /20

(13) for this finding Relations, construct congruence (Non Intersect)

No geodesic
No H.S.O



Let these congruence have y_a = constant

& if I select $\{y_a\}$ at one pt in H.S

of one H.S. It will carry to all H.S. at that pt. on H.S carried by Curve
Now do this for every curve $\{y^{a1}, y^{a2}, y^{a3}\}$

(14) Take $t(x^\alpha)$ as parameterization of Curve

Tangent vector field $= t^\alpha$

$\therefore$ Disp. along Curve $dx^\alpha = t^\alpha dt$

$x^\alpha \equiv x^\alpha(y^a, t)$

$x^\alpha(y^a, t) \therefore dx^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t dy^a + \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} dt$

(15) Another way to look at it.

$\Rightarrow$ Change in $t(x^\alpha)$

$$dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha \quad \text{(True for any Displacement)}$$

Along the curve $dx^\alpha = t^\alpha dt$

$$dt = \left(\frac{\partial t}{\partial x^\alpha} t^\alpha\right) dt$$

$$\therefore \frac{\partial t}{\partial x^\alpha} t^\alpha = 1$$

t^α : tangent vector

$\partial_\alpha t$: Gradient of t fn

(16) Original coord sys $(\{x^\alpha\})$

By construction coord. sys $(\{y^a, t\})$

y^a : Selects the Curve

t : Selects the pt. where I am at Curve

And in general $x^\alpha = f(t, y^a)$: Parametric Eqn

(17)

$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha$ $\to$ moving along one of the Curve

$\left(\frac{\partial x^\alpha}{\partial y^a}\right)_t = e^\alpha_a$ = Displacements along H. S.

= Tangent vectors along H. S.

PAGE NO.:
DATE: / / 20

(13) (18) $\alpha_t e^{\alpha}_a = 0$

(19) Normal vector $\boxed{\text{Check (18)}}$

It is not Compulsory that $n^{\alpha} \parallel t^{\alpha}$
i.e. Those Curves don't have to be H.S.O.

$$\boxed{n_{\alpha} = + N \partial_{\alpha} t \quad \downarrow \text{ Lapse fn}}$$

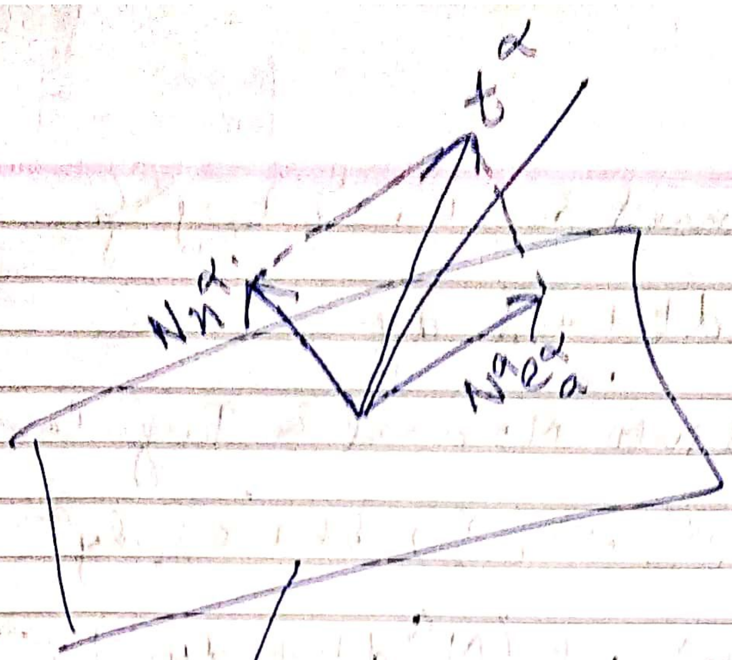
How? It is the measure of proper Distance B/w One H-S & other

(20) $n_{\alpha} e^{\alpha}_a = 0$

(21) Decompose t^{α} into $\{n_{\alpha}, e^{\alpha}_a\}$.

$$t^{\alpha} = \underbrace{N}_{\text{Lapse fn}} n^{\alpha} + \underbrace{N^a}_{\text{Shift vector}} e^{\alpha}_a$$

(22)



Proof. $\boxed{t^\alpha = N n^\alpha + N^a e^\alpha_a}$

(23) Let

$$t^\alpha = A n^\alpha + N^a e^\alpha_a \quad \text{By Def.}$$

$t^\alpha \partial_\alpha t = 1 \Rightarrow$ By Def. of Lapse for

$$n_\alpha = +N \partial_\alpha t$$

$$t^\alpha \left(-\frac{n_\alpha}{N}\right) = \left(A n^\alpha + N^a e^\alpha_a\right)\left(+\frac{n_\alpha}{N}\right)$$

$$1 = \frac{A}{N}$$

$$\therefore \underline{\underline{A = N}}$$

(24) $\therefore$ we can Define Lapse either

$$\boxed{\begin{gathered} n_\alpha = N \partial_\alpha t \\ \text{or} \\ t^\alpha = N n^\alpha + N^a e^\alpha_a \end{gathered}}$$

PAGE NO.:
DATE: / /20

(25) What is the metric in $\{t, y^a\}$

$$dx^\alpha = t^\alpha dt + e^\alpha_a dy$$

Break this into Normal & Tang. Comp.

$$= (N n^\alpha + N^a e^\alpha_a) dt + e^\alpha_a dy^a$$

$$= (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha_a$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$\boxed{ds^2 = N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt)}$$

3+1 Decomposition.

(26) For Displacement along Curve $dy^a = 0$

$$ds^2 = N^2 dt^2 + h_{ab} N^a N^b dt^2$$

Increment in proper time related to coordinate time t

$$d\tau^2 = (N^2 + h_{ab} N^a N^b) dt^2$$

Check 19.

(27) for H.S.O. $N^a = 0$

as $t^\alpha = N n^\alpha + \cancel{N^a e^\alpha_a}$

PAGE NO.:
DATE: / /20

25

$\therefore \; d\tau^2 = N^2 dt^2$ for HSO

check (9)

(28) Disp. along H.S.

$ds^2 = h_{ab}\, dy^a dy^b$.

This is what we is the Defn.

(29)

$\sqrt{-g} = N\sqrt{h}$

(30) Hamiltonian of field theory

$\mathcal{L}(\phi, \nabla_\alpha \phi)$

Earlier

$\partial_0 \varphi \longrightarrow n^\alpha \partial_\alpha \phi$

time Derivative associated with Normal

But this is not general.

assume ~~the parameter along (?)~~

fastous choice $\quad t_\alpha$ tangent along congruence

$n_\alpha \not\parallel t_\alpha$

Not general { Go along flow is more general than going along Normal

PAGE NO.:
DATE: / /20

(25) (31)

$\partial_t \phi \xrightarrow{\text{generalize to}} \mathcal{L}_{t^\alpha} \phi = \dot{\phi}$

for ϕ : scalar

$\mathcal{L}_{t^\alpha} \phi = t^\alpha \partial_\alpha \phi = \frac{d\phi}{dt}$

for scalar field $\sqrt{-g}$ won't go out but for gravity no.

(32) Canonical Momentum

Definition

$\pi = \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial \dot{\phi}}$

$\mathcal{L}(\phi, \partial_\alpha \phi)$ → scalar

$\frac{d\phi}{dt}$ = scalar

Not a scalar (due to $\sqrt{-g}$) Why?

But a Scalar Density.

(33) Hamiltonian Density

$\mathcal{H}(\phi, e^\alpha_a \partial_a \phi, \pi)$

$\dot{\phi}(\phi, e^\alpha_a \partial_\alpha \phi, \pi)$

$\mathcal{H} = \pi \dot{\phi} - \sqrt{-g}\,\mathcal{L}$ } Scalar Density.

Not a scalar

$H = \int \mathcal{H}\, d^3 y$

scalar

No $\sqrt{h}$ Because already there

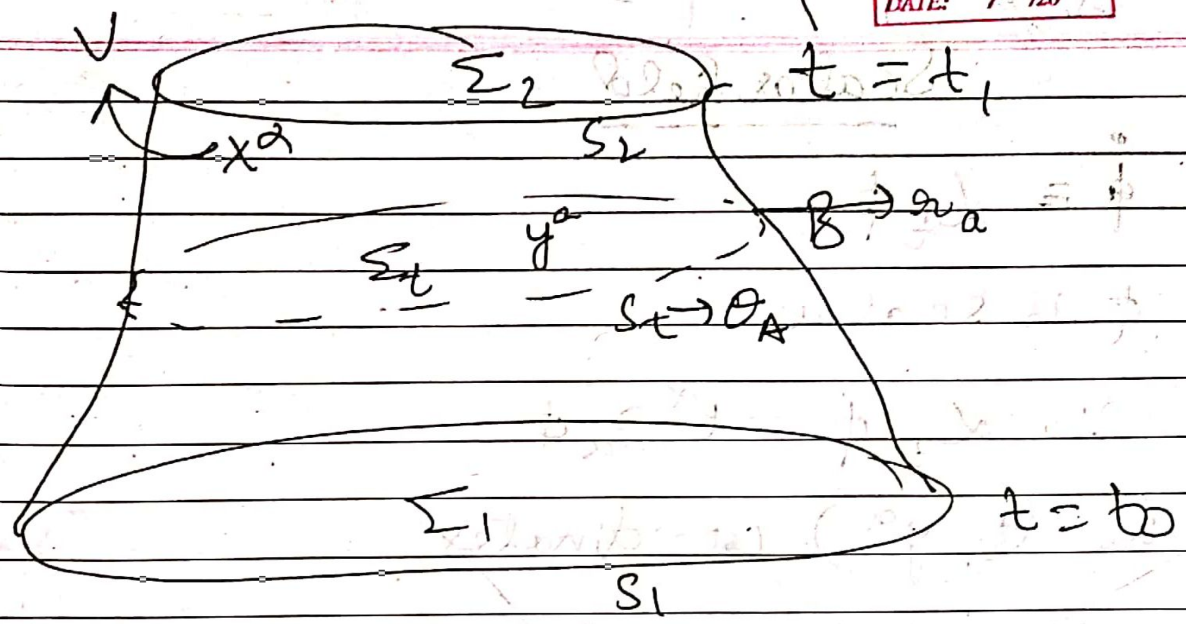
(34) GR formulation

$S_G = \int_V R \sqrt{-g}\, d^4 x + 2 \oint_{\partial V} \epsilon K \sqrt{h}\, d^3 y$

$t(x^\alpha)$

PAGE NO.:
DATE: / /20

(25)



$\partial V = \Sigma_1 + \Sigma_2 + B(\tau^i) \quad V(x^\alpha)$

V is foilated by $\Sigma_t(y^a)$

All H.S. also have Boundries S_t.

Σ_t is Bounded by $S_t(\theta_A)$

B is foliated by S_t

for Σ

$\phi(x^\alpha) = \text{const.}$

$x^\alpha(y^a)$ 4 f^n 3 par.

n_α = Normal to Σ

for S_t

$\phi(y^a) = \text{const}$

$y^a(\theta A)$ 3 f^n 2 par

r_a = Normal to S_t

$r^\alpha = r^a e^\alpha_a$

Associated 4 vector

PAGE NO.:
DATE: / /20

Curved ~~flat~~ spacetime

Scalar field

① $\dot{\phi} \equiv d_t \phi$

ϕ is scalar

$\therefore \; d_t \phi = t^\alpha \partial_\alpha \phi$

② in (t, y^a) cordinates

$t^\alpha = \left(\frac{dx^\alpha}{dt}\right) \Rightarrow t^0 = 1;$

$t^1 = \frac{dy^1}{dt}$

$$\left(\frac{dx^\alpha}{dt}\right) = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y_a} + \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \frac{dy^a}{dt}$$

$t^\alpha \partial_\alpha t = 1$

$$\left(\frac{dy^a}{dt}\right)\left(\frac{\partial \phi}{\partial y^a}\right)_t$$

③ $\frac{\partial \phi}{\partial t}$

$d_t \phi = t^\alpha \partial_\alpha \phi$

$$\frac{dx^\alpha}{dt} = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y_a} + \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \frac{dy^a}{dt}$$

PAGE NO.:
DATE: / /20

Along the curve

$$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = \frac{dx^\alpha}{dt}$$

in (t, y^a).

$$t^\alpha = \frac{dx^\alpha}{dt} = (1, 0, 0, 0)$$

$$\mathcal{L}_t \phi = \frac{\partial \phi}{\partial t}.$$

$$\left(\frac{dy^a}{dt}\right)\left(\frac{\partial \phi}{\partial y^a}\right)_t = 0$$

$$\therefore \boxed{\frac{dx^\alpha}{dt} = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a}.}$$

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

4) $\dfrac{\partial \phi}{\partial y^a} \equiv \dfrac{\partial \phi}{\partial \cancel{x}^a} e^\alpha_a$ Spatial Derivative

5) $L(q, \dot{q}, t) \longrightarrow \mathcal{L}(\phi, \mathcal{L}_t\phi, \partial_\alpha \phi e^\alpha_a)$

Density

$p = \dfrac{\partial L}{\partial \dot{q}}$ $\qquad \pi = \dfrac{\partial (\mathcal{L}\sqrt{-g})}{\partial (\mathcal{L}_t \phi)}$

$\dot{q}(p, q)$ $\qquad \overset{a}{\phi}(\phi, \partial_\alpha \phi, \pi)$

$H = p\dot{q} - L$ $\qquad \mathcal{H} = \pi \mathcal{L}_t \phi - \mathcal{L}\sqrt{g}$

$H(p, q)$ $\qquad \mathcal{H}(\phi, \partial_\alpha \phi, \pi)$

PAGE NO.:
DATE: / /20

25 3 (6) Lagrangian is Scalar.
Hamiltonian is also scalar.

Here

Hamiltonian is not a Scalar.

32 (7) $\mathcal{H} = \sqrt{h}\, H_{scal} = \frac{\sqrt{-g}}{N} H_{scal}$

$$H(p,q) = \int_{\Sigma_t} \mathcal{H}\, d^3y$$

functional

(8) $A = \int \mathcal{L}\, d^4x$

$$= \int (\pi\, \partial_t \phi - \mathcal{H})\, d^4x$$

32 $$= \int_{t_1}^{t_2} dt \int_{\Sigma_t} (\pi\, \partial_t \phi - \mathcal{H})\, d^3y$$

(9) $$\int_{t_1}^{t_2} dt \frac{d}{dt} \int_{\Sigma_t} (\pi \delta \partial_t \phi - \mathcal{H})\, d^3y$$

34 $$\int_{t_1}^{t_2} d\left(\int_{\Sigma_t} (\pi \delta \partial_t \phi - \mathcal{H})\, d^3y\right)$$

$$\int_{x_1}^{x_2} dx$$

$x_2 - x$

$$\int_{\Sigma_2} (\pi \delta \partial_t \phi - H)\, d^3y - \int_{\Sigma_1} (\pi \delta \partial_t \phi - H)\, d^3y$$

(16) $\int_V (\partial_\alpha A^\alpha)\, d^4x = \int_{\partial V} A^\alpha\, d\Sigma_\alpha \qquad d\Sigma_\alpha = d^3x\, n_\alpha$

PAGE NO.:
DATE: / /20

(25) $\int_V (\nabla_\alpha A^\alpha) \sqrt{-g}\, d^4x = \int_{\partial V} A^\alpha\, d\Sigma_\alpha$

$d\Sigma_\alpha = \sqrt{\pm h}\, d^3y \cdot n_\alpha$

in 3D $\int_V \partial_\alpha A^\alpha\, d^3y = \int_{\partial V} A^\alpha\, dS_\alpha \qquad dS_\alpha = r_\alpha\, d^2\theta$

$\int_V \nabla_\alpha A^\alpha \sqrt{h}\, d^3y = \int_{\partial V} A^\alpha\, dS_\alpha \qquad dS_\alpha = r_\alpha \sqrt{\sigma}\, d^2\theta$

$\frac{\partial H_s}{\partial(\partial_a q)}\, \delta(\partial_a q) \sqrt{h}\, d^3y$

$\int \partial_a \left(\frac{\partial H_s}{\partial(\partial_a q)}\, \delta q \right) \sqrt{h}\, d^3y - \int \partial_a \left(\frac{\partial H_s}{\partial(\partial_a q)} \right) \delta q \sqrt{h}\, d^3y$

(32) $\partial_a \phi = \frac{\partial \phi}{\partial y^\alpha}\, e^\alpha_a$

$\nabla_a \phi \equiv e^\alpha_a\, \partial_\alpha$

$\nabla_a A^b_b = \nabla_\alpha A^\phi_\beta\, e^\alpha_a\, e^\beta_b$

(34) $\nabla_a \phi = \nabla_\alpha \phi\, e^\alpha_a := \partial_\alpha \phi\, e^\alpha_a$

$\int \nabla_a \left(\frac{\partial H_s}{\partial(\partial_a q)}\, \delta q \right) \sqrt{h}\, d^3y - \int \nabla_a \left(\frac{\partial H_s}{\partial(\partial_a q)} \right) \delta q \sqrt{h}\, d^3y$

PAGE NO.:
DATE: / /20

$$x_{\bar{\mu}} = \underbrace{\eta_{\bar{\mu}\alpha} L^{\alpha}{}_{\bar{\beta}}\, \eta^{\bar{\beta}\gamma}}\, x_{\gamma}$$

$$\frac{\partial x_{\nu}}{\partial x_{\bar{\mu}}} = (\eta L \eta)^{-1}$$

$$(\Lambda^{-1})^{\alpha}{}_{\beta} = \Lambda_{\beta}{}^{\alpha}$$

$$= \Lambda_{\beta\gamma}\Lambda^{\gamma\alpha}$$

$$= \Lambda$$

$$\Lambda_{\beta}{}^{\alpha} = \Lambda_{\beta\lambda}\Lambda$$

$$\Lambda_{\beta\alpha} = \Lambda_{\beta}{}^{\gamma}\Lambda^{\alpha} \qquad \Lambda_{\beta}{}^{\alpha}\Lambda^{\beta}{}_{\alpha}$$

$$\Lambda_{\beta\alpha} = \Lambda_{\beta\gamma}\Lambda^{\gamma}{}_{\alpha}$$

$$\Lambda_{\beta}{}^{\alpha} = \Lambda_{\beta\gamma}\Lambda^{\gamma\alpha}$$

$$\Lambda_{\beta}{}^{\alpha} = \Lambda_{\beta\delta}\Lambda^{\delta}{}_{\gamma}\Lambda^{\gamma\alpha}$$

$$\Lambda_{\beta}{}^{\alpha} = \Lambda_{\beta\delta}\Lambda^{\gamma\alpha}\Lambda^{\delta}{}_{\gamma}$$

∇

∂x^2

x_1

$x_2 - x$

PAGE NO.:
DATE: / /20

(3) 3+1 formulation matter action also if I have matter.

L-16

PAGE NO:
DATE: / /20

(1) Σ_t: $\begin{cases} t(x^\alpha) = \text{const} \\ x^\alpha = x^\alpha(y^a) \quad \text{4 fn} - 3 \text{ par.} \\ h_{ab} = g_{\alpha\beta}\, e^\alpha_a\, e^\beta_b. \\ g^{\alpha\beta} = \epsilon n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b \\ n_\alpha \propto \partial_\alpha t \\ e^\alpha_a = \dfrac{\partial x^\alpha}{\partial y^a} \\ K_{ab} = \nabla_\alpha n_\beta\, e^\alpha_a\, e^\beta_b \end{cases}$

(2) S_t & Embedded in 3D Σ_t $\begin{cases} \phi(y^a) = \text{const}. \\ y^a = y^a(\theta^A) \quad 3 \text{ fn} - 2 \text{ par} \\ \sigma_{AB} = h_{ab}\, e^a_A\, e^b_B \\ h^{ab} = \epsilon\, r^a r^b + \sigma^{AB} e^a_A e^b_B \\ r_a \propto \partial_a \Phi \\ e^a_A = \dfrac{\partial y^a}{\partial \theta^A} \end{cases}$

2 Dim. Ext. Curv. $\Leftarrow K_{AB} = \nabla_a r_b\, e^a_A e^b_B$

(3) We can also embed it in 4D

2D embeds in 4D. spacetime.

PAGE NO.:
DATE: / /20

S_t embed in Spacetime.

(4) $\varphi(x^\alpha) = \text{const}$.

$\Rightarrow x^\alpha = x^\alpha(\theta^A)$ 4 f^n 2 par.

$= x^\alpha(y^a) = x^\alpha(y^a(\theta^A))$

$\Rightarrow e^\alpha_A = \frac{\partial x^\alpha}{\partial \theta^A} = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial y^a}{\partial \theta^A}$

$= e^\alpha_a e^a_A$ = Projecting e^α_a to S_t

3D. Tangent vector on Σ_t

$\Rightarrow r^\alpha = r^a e^\alpha_a$

Push forward.

$\Rightarrow \sigma_{AB} = h_{ab} e^a_A e^b_B$

$= (g_{\alpha\beta} e^\alpha_a e^\beta_b) e^a_A e^b_B$

$= g_{\alpha\beta} e^\alpha_A e^\beta_B$

$\Rightarrow \nabla_a r_b = \nabla_\alpha r_\beta e^\alpha_a e^\beta_b$

$K_{AB} = \nabla_a r_b e^a_A e^b_B$ $\sigma^{AB} K_{AB} = K$

$= \nabla_\alpha r_\beta e^\alpha_A e^\beta_B$

2Dim. Extrins

Completeness Reln

$\Rightarrow$

$g^{\alpha\beta} = n^\alpha n^\beta + r^\alpha r^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$.

Scanned with CamScanner

(5) Σ embedded in spacetime

$\Rightarrow \quad \Phi(x^\alpha) = 0$

$x^\alpha = x^\alpha(z^i) \qquad i\ j\ k.$

$\Rightarrow \quad n_\alpha \propto \partial_\alpha \Phi$

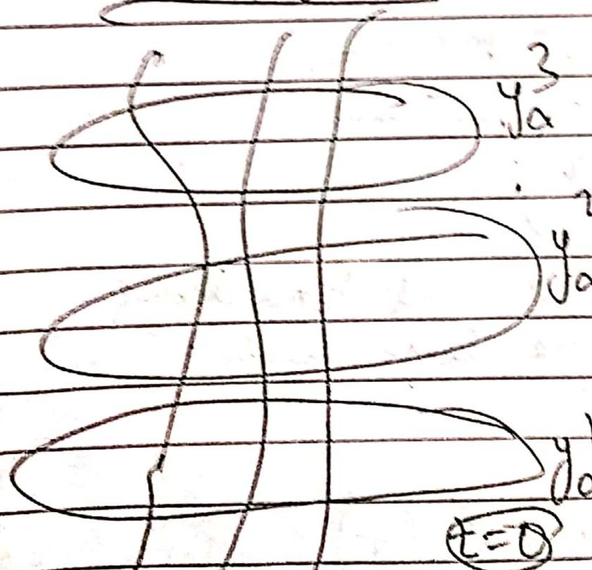
$$e^\alpha_i = \frac{\partial x^\alpha}{\partial z^i}$$

$\Rightarrow \quad \gamma_{ij} = g_{\alpha\beta}\, e^\alpha_i\, e^\beta_j$

$K_{ij} = \nabla_\alpha n_\beta\, e^\alpha_i\, e^\beta_j \qquad \gamma^{ij} K_{ij} = K$

$\Rightarrow \quad g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + \gamma^{ij} e^\alpha_i e^\beta_j$

(6) Foliation of Σ by Σ_t.



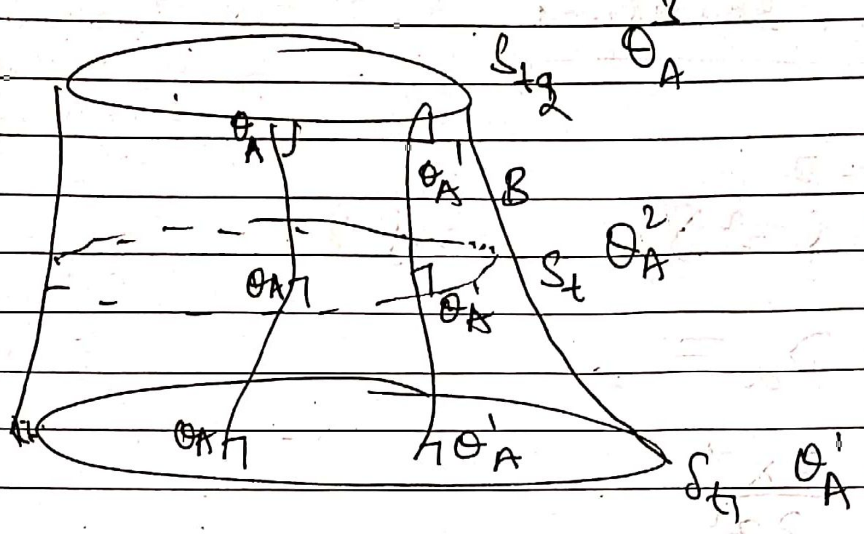
Relate y_a by setting condition on congruences.

$$t^\alpha = \frac{\partial x^\alpha}{\partial t} \quad \text{along curve}$$

$$t^\alpha = \frac{dx^\alpha}{dt}$$

$$dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha = \left(\frac{\partial t}{\partial x^\alpha} t^\alpha\right) dt$$

Here



Relating θ_A^1 θ_A^2 θ_A^3.
By Congruence on Boundary.

i.e. New long. of Curves on B along which θ_A is constant along the curve.

Restriction: Assume H.S.O.

Congruen Orthog. to S_t

$\therefore N^a = 0$ shift vector $= 0$

$(t, \theta_A) \Longrightarrow (z^i)$

$B_t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{\theta_A}$ = Tangent vector at Boundary along the Congruence.

$B_t^\alpha = N n^\alpha$ (zero shift)

$n_\alpha = N \partial_\alpha t$

PAGE NO.:
DATE: / /20

(3) ~~[illegible]~~

as

$(t, y^a) \equiv (x^\alpha)$

Thus $(t, \theta_A) \equiv (z^i)$

Disregarding x^α & z^i
Working in (t, y^a) & (t, θ_A) coord.

(10)

Choosing $z^i = (t, \theta^A)$

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{\theta_A} dt + \left(\frac{\partial x^\alpha}{\partial \theta_A}\right) d\theta^A$$

Displacement on B

$$= {}^B t^\alpha dt + e^\alpha_A d\theta^A$$

$$= N n^\alpha dt + e^\alpha_A d\theta^A$$

$n_\alpha e^\alpha_A = 0$

ds^2

Earlier Today dx^α Displacement in Spacetime

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} dt + \left(\frac{\partial x^\alpha}{\partial y^a}\right) dy^a$$

$$= t^\alpha dt + e^\alpha_a dy^a$$

$$ds = (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha_a$$

$ds^2 =$

PAGE NO.:
DATE: / /20

But Now Displacement on $\mathcal{B}$.

$$ds^2 = g_{\alpha\beta}(N n^\alpha dt + e^\alpha_A d\theta^A)(N n^\beta dt + e^\beta_B d\theta^B)$$

$$ds^2 = N^2 dt^2 + \sigma_{AB}\, d\theta^A d\theta^B$$

$$\boxed{\gamma_{ij}\, dz^i dz^j = N^2 dt^2 + \sigma_{AB}\, d\theta^A d\theta^B}$$

(11) $\sqrt{\gamma} = N\sqrt{\sigma}$ $\;\Big|\; g^{\alpha\beta}\partial_\alpha t\, \partial_\beta t \overset{*}{=} g^{tt} = \frac{\mathrm{cf}(g_{tt})}{g} = \frac{\sigma}{\gamma}$, $\quad g^{\alpha\beta}\frac{n_\alpha n_\beta}{N^2}$

(12) $$16\pi S_G = \int_V {}^4R\sqrt{-g}\, d^4x + 2\int_{\Sigma_2} K\sqrt{h}\, d^3y$$

$$\Rightarrow \quad - 2\int_{\Sigma_1} K\sqrt{h}\, d^3y - 2\int_{\mathcal{B}} \mathcal{K}\sqrt{-\gamma}\, d^3z$$

due to $\in$ timelike surface

$\Rightarrow$ Spacelike H.S. Vol $\quad h_{ab} \quad |h_{ab}| = |h|$

Timelike H.S. Vol $\quad \gamma_{ij} \quad |\gamma_{ij}| = -\gamma$

Spacetime Vol. $\quad g_{\alpha\beta} \quad |g_{\alpha\beta}| = -g$

why?

(13) Now Bulk Integral

$$\int {}^4R\sqrt{-g}\, d^4x = \int_V ({}^3R + K^{ab}K_{ab} - K^2) N\sqrt{h}\, dt\, d^3y$$

$$- 2\oint_{\partial V} (n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta)\, d\Sigma_\alpha$$

Scanned with CamScanner

(4)

PAGE NO.:
DATE: / /20

Combining all

$$16\pi S_q = \int_{t_1}^{t_2} dt \int_{\Sigma_t} ({}^3R + K^{ab}K_{ab} - K^2)\, N\sqrt{h}\, d^3y$$

$$- 2\int_{B} \left({}^{B}K + \nabla_\beta r_\alpha n^\alpha n^\beta \right) \sqrt{-\gamma}\, d^3 z$$

$$\gamma^{ij}\, {}^{B}K_{ij} = {}^{B}K = \left(\nabla_\beta r_\alpha e^\alpha_i e^\beta_j \right)\gamma^{ij}$$

$$= \nabla_\beta r_\alpha \left(\gamma^{ij} e^\alpha_i e^\beta_j \right) = \nabla_\beta r_\alpha \left(g^{\alpha\beta} + r^\alpha r^\beta \right)$$

$$-2\int_B \left(\nabla_\beta r_\alpha \underbrace{\left(g^{\alpha\beta} + r^\alpha r^\beta + n^\alpha n^\beta \right)}_{\text{from (4) completeness Reln}} \sqrt{-\gamma}\, d^3 z \right.$$

$$-2\int_B \nabla_\beta r_\alpha\, \sigma^{AB} e^\alpha_A e^\beta_B \sqrt{-\gamma}\, d^3 z$$

$$-2\int_B \sigma^{AB} K_{AB} \rightarrow \text{2 Dim. Ext.}$$

$$-2\int_B K \rightarrow \text{Trace}$$

$$\sqrt{-\gamma} = N\sqrt{\sigma}$$

$$-2\int_B {}^2K \sqrt{-\gamma}\, d^3 z$$

$$= -2\int_{t_1}^{t_2} dt \oint_{S_t} {}^2K N\sqrt{\sigma}\, d^2\theta$$

$\partial V = \Sigma_1 \cup \Sigma_2 \cup B$

On Σ_2 $\quad d\Sigma_\alpha = n_\alpha \sqrt{h}\, d^3y$

On Σ_1 $\quad d\Sigma_\alpha = -n_\alpha \sqrt{h}\, d^3y$

keeping same n_α.

on B $\quad dS_\alpha = \underset{\text{due to } \epsilon}{-} r_\alpha \sqrt{-\gamma}\, d^3z$.

Σ_2 $\quad 2 \int_{\Sigma_2} (\cancel{n_\alpha n^\beta \nabla_\beta n^\alpha} + \nabla_\beta n^\beta) \sqrt{h}\, d^3y$

$\quad 2 \int_{\Sigma_2} K \sqrt{h}\, d^3y$

$- 2 \int_{\Sigma_2} K \sqrt{h}\, d^3y$

Cancel Out

Σ_1 also Cancel Out

B $\quad n_\alpha r^\alpha = 0$

$2 \int_B (n^\beta \nabla_\beta n^\alpha - \cancel{n^\alpha \nabla_\beta n^\beta})(r_\alpha) \sqrt{-\gamma}\, d^3z$

$= -2 \int_B \nabla_\beta r_\alpha n^\beta n^\alpha \sqrt{-\gamma}\, d^3z$.

PAGE NO.:
DATE: / /20

$$16\pi S_G = \int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} \left({}^3R + K^{ab}K_{ab} - K^2 \right) N\sqrt{h}\, d^3y - \oint_{S_t} {}^2k N \sqrt{\sigma}\, d^2\theta \right\}$$

Indep. of 3-dim K_{ab}

$\Rightarrow p^{ab} \sim \dfrac{\partial \mathcal{L}}{\partial \dot{h}_{ab}}$

$\dot{h}_{ab} \sim \mathcal{L}_t h_{ab} \sim K_{ab}$.

$\therefore$ Boundary term doesn't contribute to Canonical momenta

$$\Rightarrow H = \underbrace{p^{ab}\dot{h}_{ab}}_{\text{Only Bulk term}} - \mathcal{L}$$

Bulk + Boundary

L-17

1. Till Now Shift vector doesn't appear.
2. $N, N^a \equiv$ Lagrange Multiplier

They are providing the freedom of specifying the coordinate in spacetime.

But h_{ab} is the Dynamical variable.

PAGE NO.:
DATE: / /20

(3) How to check what is the Canonical/Dyn variable?

Pick the terms in action which depend on time derivatives, ~~of~~

(4) Our action depends on $K^{ab}K_{ab}$, K^2

K_{ab}

But we know $K_{ab} \propto \mathcal{L}_t h_{ab}$

$\therefore$ Canonical Momentum would be found for conjugate h_{ab}.

(5) But we also see dependence of Action on N

But not on $\dot{N}$

$\therefore$ No conjugate momentum associated with N

$\therefore$ N is Not dynamical variable as time derivative of N is not there

(6) $\dot{N}$ plays the role of Legendre Multiplier

(7) $$\dot{h}_{ab} \equiv \mathcal{L}_t h_{ab} = \mathcal{L}_t \left(g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}\right)$$

$$= \left(\mathcal{L}_t g_{\alpha\beta}\right) e^{\alpha}_{a} e^{\beta}_{b}$$

$$= \left(\nabla_\beta t_\alpha + \nabla_\alpha t_\beta\right) e^{\alpha}_{a} e^{\beta}_{b}$$

PAGE NO.:
DATE: / /20

$t^\alpha = N n^\alpha + \underbrace{N^a e_a^\alpha}_{N^\alpha}$ → push forward

(3) $= N n^\alpha + N^\alpha$

$\nabla_\beta t^\alpha$

$\dot{h}_{ab} = 2N K_{ab} + \nabla_b N_a + \nabla_a N_b$

If Only moving normal then $\dot{h}_{ab} = 2N K_{ab}$.

Now also moving along flow (normal + Tagent to H.S) — K_{ab}, N_a

(8) Put K_{ab} in terms of $\dot{h}_{ab}$
& find p^{ab}

$$p^{ab} = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial \dot{h}_{ab}}$$

Only Bulk term would contribute
Bec. $\dot{h}_{ab} \sim K_{ab}$ & K_{ab} appers in Bulk only.

$$\therefore\ p^{ab} = \frac{\partial(\mathcal{L}_{Bulk}\sqrt{-g})}{\partial \dot{h}_{ab}}$$

$\dot{h}_{ab} = 2N K_{ab} + \nabla_a N_b + \nabla_b N_a$

~~But Bulk term doesn't contain N^a~~

$\therefore$ Take derivative w.r.t. K_{ab}

PAGE NO.:
DATE: / /20

$$p^{ab} = \frac{\partial (\mathcal{L}_{ADM} N\sqrt{h})}{\partial N \partial k_{ab}}$$

$$= \frac{1}{2\cancel{N}} \frac{\partial}{\partial k_{ab}} \left(\cancel{N}\sqrt{h} \left({}^3R + k^{ab} k_{ab} - k^2 \right) \right)$$

$$= \frac{\sqrt{h}}{2} \frac{\partial}{\partial k_{ab}} \left({}^3R + (k_{ab} k_{cd}) (h^{ca} h^{db} - h^{ab} h^{cd}) \right)$$

$$\frac{\partial}{\partial k_{ab}} {}^3R = 0.$$

$$= \frac{\sqrt{h}}{2} \left[\frac{\partial}{\partial k_{ab}} (k_{ef} k_{cd}) \right] \left(h^{ce} h^{df} - h^{ef} h^{cd} \right)$$

$$= \frac{\sqrt{h}}{2} \left[\delta^a_e \delta^b_f k_{ef} + k_{ef} \delta^a_c \delta^b_d \right]$$

$$\boxed{p^{ab} = \frac{\sqrt{h}}{2} \left(k^{ab} - k h^{ab} \right)}$$

Express k_{ab} in terms of p_{ab} & h_{ab}.

Through this s.t. $H(q, p)$

PAGE NO.:
DATE: / /20

$$\mathcal{H} = p^{ab}\dot{h}_{ab} - \mathcal{L}$$

$$= \sqrt{h}\left(K^{ab} - K h^{ab}\right)\left(2NK_{ab} + \nabla_a N_b + \nabla_b N_a\right)$$

$$- N\left({}^3R + K^{ab}K_{ab} - K^2\right)\sqrt{h}$$

$$+ 2\oint_{S_t} N^2 k \sqrt{\sigma}\, d^2\theta$$