Vector Analysis 1.

L-1 Paddy's GR

(1) Weinberg → (1) ~~Old book~~ Gravitation → Approach not correct.

(2) ~~New Book~~ Cosmology → far too detailed.

(2) ~~Event~~ → $P(t, x^1, x^2, x^3)$

(3) Dimensions of x^1, x^2, x^3 → need not be the same.
eg. (r, θ, ϕ)

(4) $(x, y, z) \rightarrow (x^5, y, z)$
↳ This is also a valid coordinate system
i.e. The transformation is valid.

But
~~$(x, y, z) \rightarrow (x^2, y, z)$ is not valid as all negatives would be mapped to positive. & ∴ for one x there could be two possibilities~~
∴ Not valid Transformation.

(5) ~~Norm~~
$$\|v\| = \sqrt{g(v,v)} = \sqrt{x^2+y^2+z^2}$$
$$\|Tv\| = \sqrt{g(Tv, Tv)} = \sqrt{x^2+y^2+z^2}$$

(6) ∴ If you change your coordinates in a funny way, pythogas theorem depends on the coordinate we choose.
↳ How to see this in Linear Algebra way?

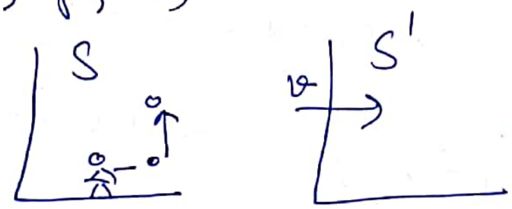
Similarly $r^2+\theta^2+\phi^2 \neq d^2$

(7) ∴ Geometry depends on the coordinate we are using.

↳ Geometry is already there coordinates are an manifestation.

(8) (x, y, z)

if



then from S' if I have to measure balls motion & since S' is also moving ∴ at particular moment where S' is i.e. at particular time t.

i.e. (x, y, z) would depend on time.

i.e. $(x(t), y(t), z(t))$

(9) Coordinates are just markers.
∴ In Physics there shouldn't be a preferred coordinate system over other.
Except for mathematically convinience.

(10) Inertial frame

Imp: In GR we drop the assumption of inertial frame being some special frame & develop physics laws which are covariant everywhere.

Cartesian Coordinate System (x, y, z) is choosen.
we could have choosen others also but for convenience.

(11) Assuming Inertial frame ∃.

(12) Inertial frame: where particles move at uniform velocity when far away from forces of any other particle.

(13) Axiom of Inertial frame: (1) Any frame in uniform motion relative to the inertial frame is itself inertial. Conversely, a frame not in uniform motion relative to inertial frame is not inertial. (2) If S' in uniform motion with velocity $\vec{v}$ relative to S then velocity of S w.r.t. S' is $-\vec{v}$.

(14) Two Postulates of STR.

(i) ~~Assuming Action Principle holds true~~ ⇒ All laws of Physics are invariant in inertial frames. i.e. Result of any exp. performed by an observer do not depend on his speed relative to other observer.

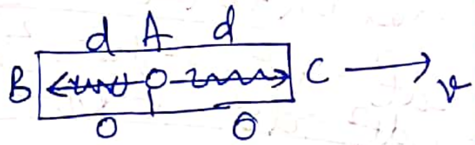
(~~By defn. of Inertial frames~~ ⇒ ∴ All Inertial frames are equivalent.

(ii) ∃ max. finite speed limit at which info can be transferred.

By Maxwell's theory, light has this max. speed which is c.

(15) Invariance of speed of light in all Inertial frames.

→ As Maxwell theory is valid in all inertial frames ∴ speed of light is same as c in all Inertial frames.

(16) 

If Ball is thrown it would have gained speed of source But it is the const. of speed of light ind. of source which Breaks simultaneity.

light beams travel at c & hits B & C simultaneously
(i) as it was left at same moment from A.

(ii) As it is light and according to observer on the ground, speed of light is constant in both directions according to Postulates as this bus is moving at constant velocity ∴ Inertial frame.

(iii) But if the speed is constant then it would hit B first & then later C ∴ Simultaneity Broken.

iv) If tennis Balls instead of light beams then 4. the velocity of balls changes according to Galilean Relativity for ground observer s.t. Simultaneity is not broken. [Ball would have gained c+V towards C & c−V towards B so that it approaches same time signal]

(17) "Simultaneous events" are observer dependent.

This is a big milestone, as if I watched my clock & marked when the event happened in my frame & other guy moving at v velocity also watched his clock and marked time when the event happened then we can say

"Flow of time is Relative"

In Polar coord also $ds'^2 = 0$ for speed of light as light travels in const θ, ϕ

$\therefore ds'^2 = c^2 dt^2 - dr^2 = 0$

(18) Notations: $(\underset{\text{time}}{t}, \underbrace{x^1, x^2, x^3}_{\text{length}})$

as c is constant $\therefore$ $(\underbrace{ct, x^1, x^2, x^3}_{\text{length}})$

$\equiv (x^0, x^1, x^2, x^3)$

Interval in Polar coord

$ds^2 = c\,dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$
$= c\,dt^2 - dr^2 - r^2 d\Omega^2$

(19) Interval: $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

$P(x^i)$, $A(x^i + dx^i)$ in one coordinate Inertial frame

$P(x'^i)$, $A(x^{i'} + dx^{i'})$ in other Inertial frame.

then $ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$

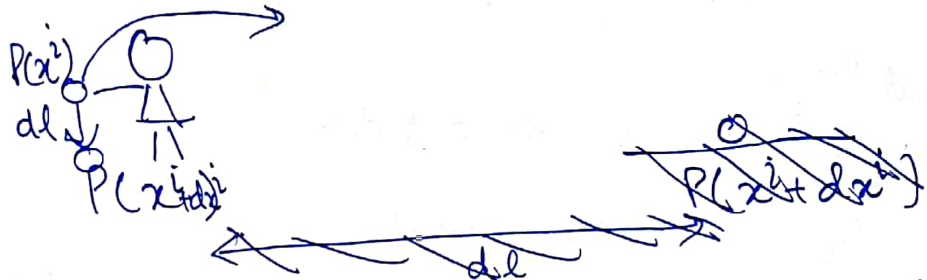
(20) Th: If $ds^2 = 0 \Rightarrow ds'^2 = 0$
If $ds'^2 = 0 \Rightarrow ds^2 = 0$ i.e. $ds^2 = 0 \Leftrightarrow ds'^2 = 0$

Scanned with CamScanner

... Charge is $\rho = 0$ at Boundary

Proof: $ds^2 = 0 \Rightarrow c^2 = \frac{dl^2}{dt^2}$

take Eg.



if $ds^2 = c^2dt^2 - dl^2 = 0$

$\therefore$ Both events are Connected by Light ray.

$\therefore$ Seeing it like light goes from One place to other

Now $ds'^2 = c^2 dt'^2 - dl'^2$

As in Other frame c is same

$\therefore$ In this frame also, light would go from One place to another at same speed i.e. $\frac{dl'^2}{dt'^2} = c^2$

$\therefore ds'^2 = 0$

$$\boxed{ds^2 = g(d\vec{x}, d\vec{x}) = \eta_{ij}\, dx^i dx^j}$$

(21) Th. $ds^2 = ds'^2$

where $c_{ij}(\vec{q}, \dot{\vec{q}}, t) \quad \forall\, ij$

Proof:

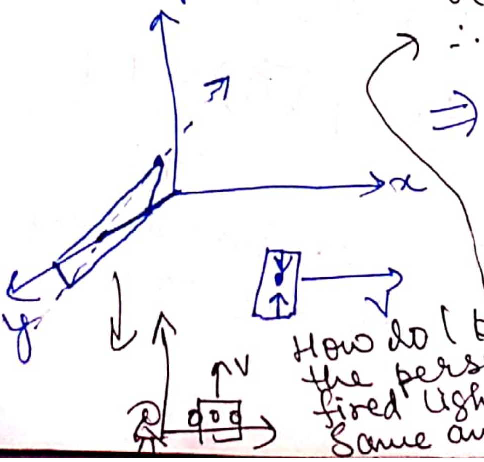
$$\begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \\ c_{30} & c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

As time is Hom. in inertial frames $\quad c_{ij}(\vec{q}, \dot{\vec{q}})$

As space is Hom. & Isotropic $\quad c_{ij}(|\dot{\vec{q}}|)$

in the S' $\leftarrow \rightarrow$ should send both at same angle

$\therefore$ in S it should



$\Rightarrow dy' = c_{22}(|\vec{v}|)\, dy$

$dz' = c_{33}(|\vec{v}|)\, dz$

How do I know the person has fired light at Same angle?

As c_{ij} shouldn't depend on the Exp. I do. They are Ind.

$\therefore dt' = c_{00}\, dt + c_{01}\, dx$

$dx' = c_{10}\, dt + c_{11}\, dx$

$$ds'^2 = (c_{00}^2 - c_{10}^2)\,dt^2 + (c_{01}^2 - c_{11}^2)\,dx^2 + 2(c_{00}c_{01} - c_{10}c_{11})\,dt\,dx \quad (1)$$
$$- c_{22}^2\,dy^2 - c_{33}^2\,dz^2$$

Let $ds'^2 = 0$

$\therefore ds^2 = 0 \Rightarrow dt = \pm dr$

Let $dx \to -dx$ in (1)

$$0 = (c_{00}^2 - c_{10}^2)\,dr^2 + (c_{01}^2 - c_{11}^2)\,dx^2 \mp 2(c_{00}c_{01} - c_{10}c_{11})\,dr\,dx$$
$$- c_{22}^2\,dy^2 - c_{33}^2\,dz^2$$

$$0 = (c_{00}^2 - c_{10}^2)\,dr^2 + (c_{01}^2 - c_{11}^2)\,dx^2 \pm 2(c_{00}c_{01} - c_{10}c_{11})\,dr\,dx$$
$$- c_{22}^2\,dy^2 - c_{33}^2\,dz^2$$

$$\therefore c_{00}c_{01} = c_{10}c_{11}$$

$$\therefore ds'^2 = (c_{00}^2 - c_{10}^2)\,dt^2 + (c_{01}^2 - c_{11}^2)\,dx^2 - c_{22}^2\,dy^2 - c_{33}^2\,dz^2 \quad (2)$$

& $(c_{00}^2 - c_{10}^2)\,dr^2 = (c_{11}^2 - c_{01}^2)\,dx^2 + c_{22}^2\,dy^2 + c_{33}^2\,dz^2$

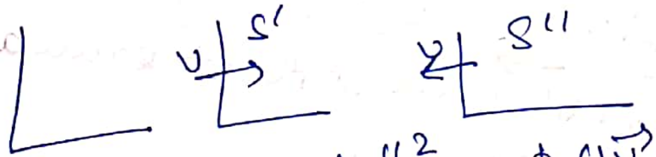
$\therefore c_{00}^2 - c_{10}^2 = c_{11}^2 - c_{01}^2 = c_{22}^2 = c_{33}^2$

$\therefore$ in (2)

$$ds'^2 = c_{22}^2\,dt^2 - c_{22}^2(dx^2 + dy^2 + dz^2)$$

$ds'^2 = c_{22}^2\,ds^2$ where $c_{22}(|\vec{v}|)$

$ds'^2 = \phi(|\vec{v}|)\,ds^2$



$$ds''^2 = \phi(|\vec{v}|)\,ds'^2 = \phi^2(|\vec{v}|)\,ds^2$$

But $ds''^2 = ds^2$ By Ax. of In. frame $\quad \phi^2(|\vec{v}|) = 1 \quad \therefore \phi(|\vec{v}|) = \pm 1$

$c_{22}^2 = \pm 1$

It can't imaginary $\therefore c_{22} = 1$

$\therefore$ $\left.\begin{array}{l} dy' = dy \\ dz' = dz \end{array}\right\} \Rightarrow \begin{array}{l} y' = y \\ z' = z \end{array}$

(22) $ds^2 = \sum_{a,b=0,3} \eta_{ab}\, dx^a dx^b = \eta_{ab}\, dx^a dx^b = g(d\vec{x}, d\vec{x})$

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(23) $ds^2 = 0$ Null

< 0 Space like

> 0 Timelike.

$g(d\vec{x}, d\vec{x}) = 0$

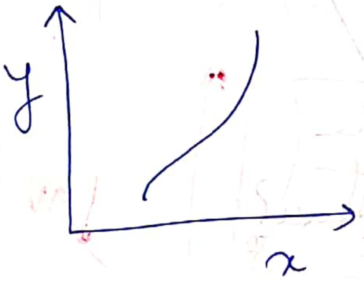
More generally for Any vector $\vec{A}$

$g(\vec{A}, \vec{A}) = 0$ Null

$g(\vec{A}, \vec{A}) < 0$ space

$g(\vec{A}, \vec{A}) > 0$ Time

(24) Physical Interpretation of ds^2



$y = f(x)$
Bijection
$\therefore$ Inverse $\exists$

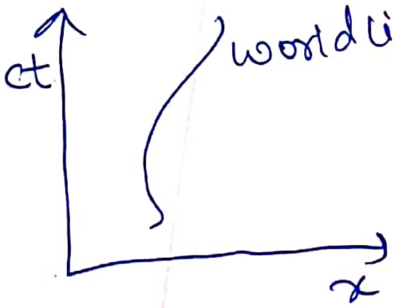
But, if

$y = f(x)$
Not Bijective
$\therefore$ Inverse $\nexists$

But we want x fn to be bijective
$\therefore$ ~~But~~ we can parameterize the curve

$y = f_1(s)$ where s is the length of the arc.
$x = f_2(s)$

(25)

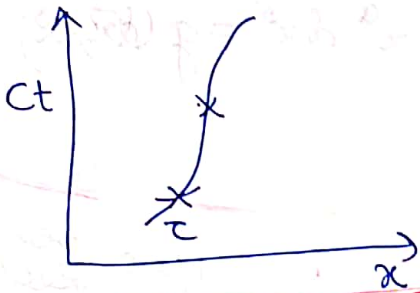


The clock of the moving guy ~~tells~~ parameterizes as it tells where the guy is.

$\therefore \tau =$ Proper time

$t(\tau), x(\tau)$

(26)



$$ds^2 = c^2dt^2 - dx^2 = c^2d\tau^2$$

$$d\tau^2 = dt^2\left(1 - \frac{1}{c^2}\frac{dx^2}{dt^2}\right)$$

$$d\tau^2 = dt^2\left(1 - \frac{v^2}{c^2}\right)$$

$$d\tau = dt\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\tau = \int\left(1 - \frac{v^2}{c^2}\right)^{1/2} dt$$

Arc length here denotes the total time that has elapsed in a moving clock.

By def. $\boxed{c^2d\tau^2 = dl^2 \quad c^2ds^2 = -dl^2}$ → Proper Dist.

$v(t)$ can be a f^n of time here i.e. frame can be accelerated.

(27) Corollary: Moving clock would move slowly.

$$d\tau = dt\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

What is the meaning of proper Distance?

(28) $ds = c\,d\tau$

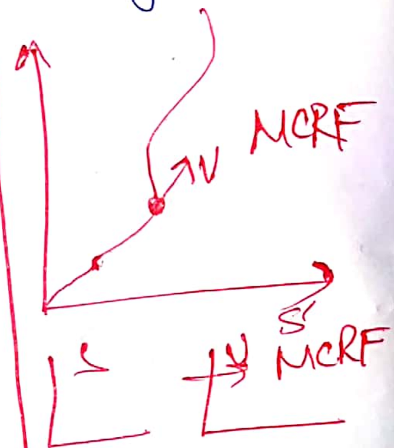
$S = c\tau$

$\therefore \quad s = c\int\left(1 - \frac{v^2}{c^2}\right)^{1/2} dt = s'$

Physical Interpretation?

Reason Physically why $ds^2 = ds'^2$?

Here in the accelerated frame we have used $ds^2 = ds'^2$?



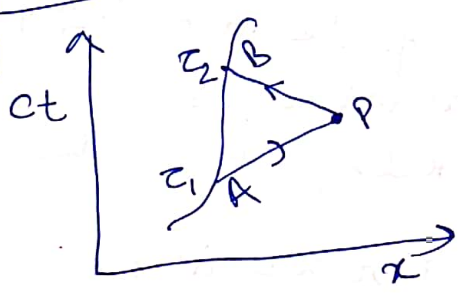
$ds'^2 = ds^2$

as I have taken arbitrary

$\therefore$ valid for whole path

$\therefore \quad d\tau = \frac{dt}{\gamma}$

(29)



$$t' = \frac{\tau_2+\tau_1}{2}$$
$$x' = \frac{c(\tau_2-\tau_1)}{2}$$
(*)

Here I am taking speed of light c even in Non inertial frame?

let for the traj. $t_0 = f_0(\tau)$
$x_0 = f_1(\tau)$

Answer: Put Inertial Observer moving at speed v

at $t \to t+dt$ its speed equal to Arbitrary observer. He sees light speed of c. Then put another observer moving at $v+dv$ at $t+dt \to t_1$ & do same

for that time $t \to t+dt$ Both frames are equivalent $\therefore$ Arbitrary observer sees velocity of speed as c

$\therefore A(f_0(\tau_1), f_1(\tau_1))$
$B(f_0(\tau_2), f_1(\tau_2))$

let (x,t) be of P
Now as slope of AP is c
PB is $-c$

$$\therefore \frac{x - f_1(\tau_1)}{t - f_0(\tau_1)} = c \quad \text{--- (1)}$$

$$\frac{f_1(\tau_2) - x}{f_0(\tau_2) - t} = -c \quad \text{--- (2)}$$

shouldn't $x = \gamma v t$ in SR?

(30) For constant velocity moving frame.
$\tau = \frac{t}{\gamma}$ $\therefore$ $f_0(\tau) = \tau\gamma$
$f_1(\tau) = v f_0(\tau) = v\tau\gamma$

from (1) $$f_0(\tau_1) = \frac{\left(t - \frac{x}{c}\right)}{\left(1-\frac{v}{c}\right)}$$

from (2) $$f_0(\tau_2) = \frac{\left(\frac{x}{c}+t\right)}{\left(1+\frac{v}{c}\right)}$$

Puting (*) we get
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(-vt + x)$$

(31) $x' = \gamma(x - vt)$

$t' = \gamma(t - vx)$

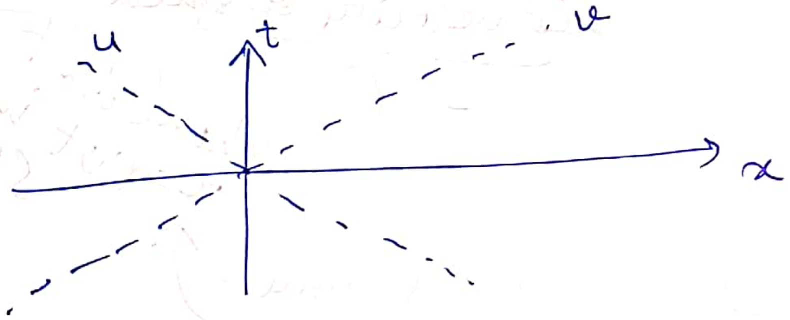
$$x' - t' = (x - t)\sqrt{\frac{1+v}{1-v}}$$

$$x' + t' = (x + t)\sqrt{\frac{1-v}{1+v}}$$

$\therefore$ good way of seeing Lorentz Transfⁿ.

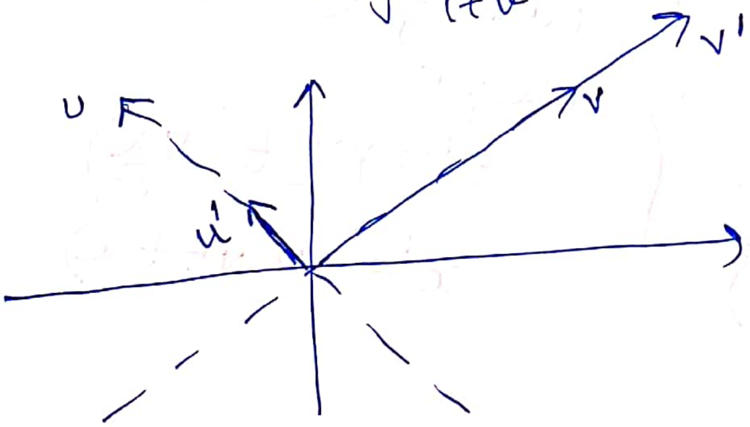
let $x - t = v$

$x + t = u$



(32) $$\underline{v}' = \underline{v}\sqrt{\frac{1+V}{1-V}}$$ Elongated v'

$$\underline{U}' = \underline{U}\sqrt{\frac{1-V}{1+v}}$$ Shrunk U'



(32) $\cancel{x^2 - t}$ $t^2 - x^2 = t'^2 - x'^2$

also comes from here.

...nt. Charge if f = 0 at Boundary, ... Then ... make it ... ⇒ Charge

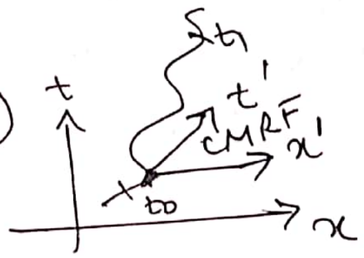
(1) Difference B/w Proper Length & proper Distance?

(1) Twin paradox — Schutz Ch-1

(2) Axis Elongated — Resonance Paper.

) Proper Distance & Proper time.

Def: $\boxed{\begin{array}{l} c^2 d\tau^2 = dl^2 \\ c^2 ds^2 = -dl^2 \end{array}}$ (for timelike) (for spacelike) (in CMRF)



$\Rightarrow c^2 d\tau^2 = c^2 dt^2 - dx^2$

$d\tau^2 = \frac{dt^2}{\gamma^2} \Rightarrow d\tau = \frac{dt}{\gamma} \Rightarrow \tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt = t'$

$\Rightarrow c^2 ds^2 = -dl^2 = dx^2 - c^2 dt^2$

$ds^2 = -dt^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow ds = i\frac{dt}{\gamma}$

Twin Paradox

There are two explainations

① Effect of different standards of simultaneity in

31 different frames.

② Acceleration.

Aging is direct effect of acceleration.

There are 2 seperate inertial frames, one way out & one way backward, this frame swith is the reason for aging difference.

⇒ Nor GR, nor acceleration is necessary to explain twin Paradox.

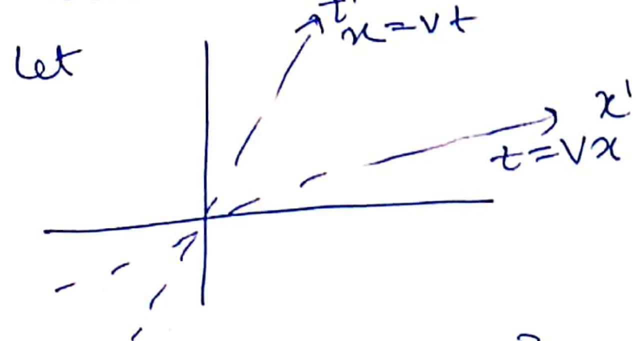
For eg. Assume a pair of observers, one travelling away from starting point & another travelling toward it, passing by each other where turnaround point would be. At this moment, the clock reading in the first observer is transfered to other, both maintaining constant speed, with both trip times being added at the end of journey.

for pt. charge if $f = 0$ at Boundary ... then to make it ... Charge

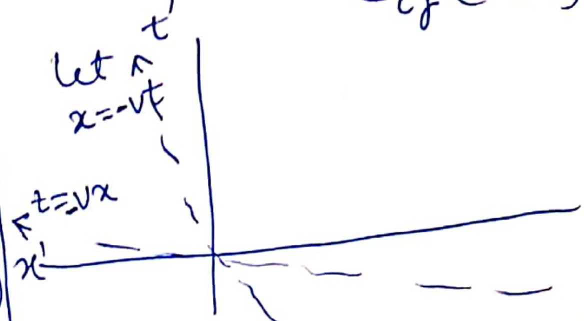
$$\therefore \begin{rcases} dt' = c_{00}\, d\vec{t} + c_{01}\, d\vec{x} \\ d\vec{x}' = c_{10}\, d\vec{t} + c_{11}\, d\vec{x} \end{rcases} \xRightarrow{\text{Intg.}} \begin{rcases} t' = c_{00} t + c_{01} x \\ x' = c_{10} t + c_{11} x \end{rcases}$$

where $c_{ij}(|\vec{v}|)$

let



let



$$c_{11}^2 - c_{01}^2 = 1 = c_{00}^2 - c_{10}^2 \quad — (*)$$

Putting $x = vt \equiv x' = 0$ what if $x = -vt$?

$\cancel{c_{00} + c_{01} v = 0}$

$$c_{10} + c_{11} v = 0 \Rightarrow \boxed{c_{10} = -c_{11} v}$$

Putting $t = vx \equiv t' = 0$

$$c_{00} v + c_{01} = 0 \Rightarrow \boxed{c_{01} = -c_{00} v}$$

$$\therefore \begin{rcases} x' = c_{11}(x - vt) \\ t' = c_{00}(t - vx) \end{rcases} — (*')$$

Putting in $(*)$ we get

$$c_{00}^2 - c_{11}^2 v^2 = c_{11}^2 - c_{00}^2 v^2 \Rightarrow c_{00}^2 = c_{11}^2$$

$\Rightarrow c_{00} = \pm c_{11}$

as $v \to 0$ $\quad x' \to x$, $t' \to t$ in $(*')$

$$\therefore \begin{rcases} c_{00} = 1 \\ c_{11} = 1 \end{rcases} \text{in } v \to 0 \quad \therefore \underline{c_{00} = c_{11}}$$

$c_{00}^2 - c_{00}^2 v^2 = 1$

$\therefore c_{00}^2 = \dfrac{1}{1 - v^2}$

$c_{00} = \pm \gamma$

But $v \to 0$, $c_{00} = 1$ $\quad \therefore \underline{c_{00} = \gamma = c_{11}} \Rightarrow$

$x = -vt \equiv x' = 0$

$\boxed{c_{10} = c_{11} v}$

$t = -vx \equiv t' = 0$

$\boxed{c_{00} v = c_{01}}$

$$\begin{rcases} t' = c_{00}(t + vx) \\ x' = c_{11}(x + vt) \end{rcases} (*)$$

Similarly

$t' = \gamma(t + vx)$
$x' = \gamma(x + vt)$
$y' = y$
$z' = z$.

$c_{00} = \gamma$
$c_{01} = \gamma v$
$c_{10} = \gamma v$
$c_{11} = \gamma$

here x^i scalar
$\therefore$ + & − sign diff due to vector

$t' = \gamma(t - vx)$
$x' = \gamma(x - vt)$
$y' = y$
$z' = z$.

11.

(32) why $y' = y$, $z' = z$?

Ans
Because if we are working in 2D only then they won't interfere. Now if we have more dimensions they won't disturb

Sir says it is tautology

(33) x' is a linear comb. of x & t.
t' is a Lin. Comb. of x & t.

(34) $x^{i'} = L^{i'}_{j} x^{j}$

As from $x' \xrightarrow[\text{linear Transf.}]{T} x$

$T(\alpha a + \beta b) = \alpha T(a) + \beta T(b)$

(35) Definition: 4 vector $q^{i} = (q^{0}, \vec{q})$

Any set of 4 qty s.t. $q^{i'} = L^{i'}_{j} q^{j}$

(36) Def: vector: which transforms like a vector

$\vec{k}$ = wave vector

$\frac{1}{\vec{k}}$ is not a vector as it doesn't transf. in a vector way.

$\therefore \lambda^{i} = \frac{2\pi}{k^{i}}$ is not a vector.

(37) $\therefore$ 4 vector is sth which transforms like $dq^{i'} = L^{i'}_{j} dq^{j}$

Not $x' = L^{i'}_{j} x^{j}$ Bec. in GR $dq^{i'} = \frac{\partial q^{i'}}{\partial x^{i}} dx^{i}$

~~covector~~ contravariant vector

(38) as $x^{i'} = L^{i'}_{j} x^{j}$
then $dx^{i'} = L^{i'}_{j} dx^{j}$

(39) $dx^{i'} = L^{i'}_{j} dx^{j}$

This def. carries forward to GR while first def. doesn't.

Bcz. as vectors under Rotation will make linear transf. of themselves.

But Transfn from $(x, y, z) \longrightarrow (r, \theta, \phi)$ is not a Linear Transfn. We have to do locally.

In the same way $dx^{i'} = L^{i'}_{j} dx^{j}$ does that locally.

$\therefore$ Def of 4 vector $\Rightarrow$ $dx^{i'} = L^{i'}_{j} dx^{j}$

(40) $\cancel{u^i = \frac{dx^i}{ds}}$ $\qquad \vec{u} = \vec{e_0'} = \frac{dx^i}{d\tau}$

This u^i is a 4 vector as dx^i will change like 4 vector & ds is invariant

$\therefore$ du^i is a 4 vector.

$\hookrightarrow$ 4 velocity

Doubt

(41) $u^0 = \frac{dt}{ds} = \gamma$

$\vec{u} = \gamma \vec{v}$

$\therefore \quad u = (\gamma, \gamma \vec{v})$

$\frac{c\,dt}{ds}$

$u^0 = \gamma c$

$\vec{u} = \gamma \vec{v}$

if calculated from $\vec{e_0'}$

$\vec{u} = L^0_{0'} e_0 + L^1_{0'} e_1$

$E = m\gamma$

$\vec{p} = \frac{m\gamma v}{c}$?

(42) $p^i = m u^i = m\gamma(1, \vec{v})$ $\qquad$ m : Rest mass

4 momentum

(43) Particles have one mass only i.e. Rest mass.

$\vec{p} = m\vec{v}$ (Newton)

$\vec{p} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} \vec{v}$ (STR)

$\therefore$ We think $m_{Rel} = \gamma m$

Charge u. f = 0 at Boundary ...

But if we have calculated relativistic mass exp. from Rel. Energy Exp. then we would have got 13. other Exp.

Similarly if we have calculated from $\frac{\text{force}}{\text{mass}} = \text{acc}$ expression, we would have got another exp. & also longitudinal mass, transverse mass.

$\therefore$ Particles have one mass i.e. Rest mass.

~~(44) Def. $u_i \equiv \eta_{ab} u^a$~~

~~$\therefore ds^2 = dx^a d$~~

(44) Definition : $q^i \longrightarrow q_j \equiv \eta_{ij} q^j$ (covariant)

Lowering of Index.

(45) as $ds^2 = \eta_{ab} dx^a dx^b = dx^a dx_a$

(46) $q_0 = q^0$

$q_1 = -q^1$

$q_2 = -q^2$

$q_3 = -q^3$

$\therefore (q_0, -\vec{q}) \equiv (q^0, \vec{q})$

$$\boxed{g(\vec{A}, \vec{A}) = A^i A_i = \eta_{ij} A^i A^j = \eta^{ij} A_j A_i = \text{Square length}}$$

(47) $u^i u_i = u^i (\eta_{ij} u^j) = \eta_{ij} u^i u^j = g(\vec{u}, \vec{u})$

$u^i u_i = \eta_{ij} u^i u^j$

$= \eta_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$ from (45)

$= \frac{ds^2}{ds^2} = 1$

or By def.

$g(\vec{u}, \vec{u}) = g(\vec{e}_0, \vec{e}_0) = 1$

Scalar Defin

$\phi(x) = \bar{\phi}(\bar{x})$

t' t x' x at same event

$A_i A^i = \text{Scalar}$

But depends on x like $\phi(x)$

But $u_i u^i = 1 = \text{constant}$

(48) Norm of 4 vector $\|\cdot\| : V \longrightarrow \mathbb{R}$

$\|u\| = u^i u_i$

(49) Therefore $\|u\| = 1$ always.

(50) We can also calculate it as

$$u^i u_i = u^0 u_0 + u^1 u_1 + u^2 u_2 + u^3 u_3$$

$$= u^{0\,2} - \sum_{i=1}^{3} u^{i\,2}$$

$$= \gamma^2 - \gamma^2 v^2$$

$$= \gamma^2 (1 - v^2)$$

$$= 1$$

$$\boxed{\begin{aligned} u^i u_i &= \eta_{ij} u^i u^j \\ &= \eta_{ij} \frac{dx^i dx^j}{d\tau^2} \\ &= \frac{ds^2}{d\tau^2} = \varepsilon \end{aligned}}$$

as. $q_i = \eta_{ij} q^j$

$$\therefore q_1 = \eta_{1j} q^j = \eta_{11} q^1 = -q^1$$

(51) ~~$g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow$~~

$$g : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$g(A, B) = \sum_i \alpha_i^* \beta_i$$

or

$$g(e_i, e_j) = \delta_{ij}$$

(52) Similarly here

$$g : M \times M \longrightarrow \mathbb{R}$$

$$g(A, B) = \eta_{ij} A^i B^j$$

or

$$g(e_i, e_j) = \eta_{ij}$$

Paddy GR

① $x^i = (t, x, y, z)$
or
$x^i = (t, r, \theta, \phi)$

② We can go further work out the properties of 4 vectors, dynamics of particle, also dynamics of particle by introducing Action principle.

③ Other way is to develop mathematical machinary. 4-Tensors etc.

④ $$L(v) = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ $[L^{i'}_{\ i}]$

$$L(-v) = L^{-1}(v) = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ $L^{i}_{\ i'}$

⑤ Def. $q_i = \eta_{ij} q^j$

$A^{i'} = L^{i'}_{\ i}(v) A^i$
$A_{i'} = L^{i}_{\ i'}(-v) A_i$

& as seen $L^{i'}_{\ i}(v)$ is inverse of $L^{i}_{\ i'}(-v)$

Norm of q 4vector $\quad q^i q_i = \eta_{ij} q^i q^j = (q^0)^2 - (|\vec{q}|)^2$

Norm of q 4vector $= \|q\| = q^i q_i = g(q^i, q^i)$

As $ds^2 = \eta_{ij} dx^i dx^j$

$\therefore$ ds^2 is norm of $d\vec{x}$ $\quad ds^2 = \|d\vec{x}\| = g(d\vec{x}, d\vec{x})$

⑥ As ds^2 is invariant we have shown that
$ds^2 = \|d\vec{x}\| = g(d\vec{x}, d\vec{x})$ $= (t, x, y, z)$
Similarly $\|q\|$ norm of any 4 vector is invariant. Bec. components of any vector transform in the same way as the coordinate.
As Before 4velocity norm is 1.

(7) if we have 2 4 vectors
A, B

then $\boxed{M^{ij} \equiv A^i B^j}$

M has 16 componenets as A^i has 4 & B^j has 4 components. General 2nd Rank Tensor T_{ik} cannot be expressed as product of 2 4-vectors

(8) M is a special kind of matrix. Not every matrix can be written as product of 2 covectors.

(9)

$$M^{i'j'} = A^{i'} B^{j'}$$
$$= L^{i'}_{i} L^{j'}_{j} A^i B^j$$
$$= L^{i'}_{i} L^{j'}_{j} M^{ij}$$

2nd Rank Tensor T_{ik} is defined to be set of $4\times4=16$ qty which transform like the product $A_i B_k$ of 2 4-vectors under L.T.

(10) 2 Rank Tensor: which transform like

$$A^{i'j'} = L^{i'}_{i} L^{j'}_{j} A^{ij}$$

1 Rank Tensor $\equiv$ 4 vector

$$A^{i'} = L^{i'}_{j} A^{j}$$

n Rank Tensor : $\underbrace{A^{i'j'k'\cdots}}_{n} = L^{i'}_{i} L^{j'}_{j} L^{k'}_{k} \cdots A^{ijk\cdots}$

(11) Th. $M^{ij} = \underbrace{\dfrac{M^{ij}+M^{ji}}{2}}_{\text{Symmetric}} + \underbrace{\dfrac{M^{ij}-M^{ji}}{2}}_{\text{Antisym.}}$

(12) S^{ij} = Symmetric

$$S^{ji} = \frac{M^{ji}+M^{ij}}{2} = S^{ij}$$

A^{ij} = Antisymmetric

$$A^{ji} = \frac{M^{ji}-M^{ij}}{2} = -A^{ij}$$

$$\boxed{S^{ij} = S^{ji} \qquad A^{ij} = -A^{ji}}$$

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⑬ Th. $\cancel{A^{ij}}$ $A_{ij}\, S^{ij} = 0$ (Contraction) 17.

Corollary. $X_{ij}\, S^{ij} = S^{x}_{ij}\, S^{ij}$
where X_{ij} is arbitrary tensor.

Proof. By (11) $X_{ij} = S^{x}_{ij} + A^{x}_{ij}$
$\therefore$ By (13) Th.
$$X_{ij}\, S^{ij} = S^{x}_{ij}\, S^{ij}$$

Corollary $X_{ij}\, A^{ij} = A^{x}_{ij}\, A^{ij}$

(14) q^{i} = Contravariant vector
A^{ij} = Contravariant 2 Rank Tensor
A_{ij} = Covariant 2 Rank Tensor
q_{i} = Covariant vector.

Mixed Indices $T^{i}{}_{j}$

(15) Scalar quantity
$\phi'(x') = \phi(x)$ like $ds'^2 = ds^2$

Eg. (i) Norm of the vector
→ in Euclidean Cord. Transf.
→ in Lorentz Transf. as in (6)
(ii) Temperature
$T(x)$ would remain same after Transfn

(16) $$\frac{\partial \phi'}{\partial x^{i'}} = \frac{\partial x^{j}}{\partial x^{i'}}\frac{\partial \phi}{\partial x^{j}} = L^{j}_{i'}\frac{\partial \phi}{\partial x^{j}}$$

Earlier we have used x' as funct. of (x,t).
Now x^{j} ~~as~~ is f^n of (x', t').
$x' = \gamma(x' - vt)$
$\therefore$ $v \longrightarrow -v$
Transformation invertible.

(17) By doing this we get: $L^{j}_{i'} \equiv L^{i'-1}_{j}$ Imp

L^{-1} has same elements as L just $v \to (-v)$

(18) $\therefore \dfrac{\partial \phi'}{\partial x^{i'}} = L^{j}_{i'}\left(\dfrac{\partial \phi}{\partial x^{j}}\right)$ ∴ It transforms like a covariant vector.

↳ Example of this is Normal to Surface.

(19) Surface

$S(x_1, x_2 \,----\, x_n) = \text{const.}$

↳ This is like scalar

e.g. $x^2+y^2+z^2 = 9$

radius $= 3$

(20) We can also take $S(x^0, x, y, z) = C$

$S(x^{0'}, x', y', z') = C$

∴ S is invariant.

S is scalar.

$A_{i'} = L^{j}_{i'}(v) A_j$

$A_i = L^{j'}_{i}(-v) A_{j'}$

eg. $\vec{e}_0$

Covariant.

$A^{i'} = L^{i'}_{j}(v) A^{j}$

$A^{i} = L^{i}_{j'}(-v) A^{j'}$

(21) Normal

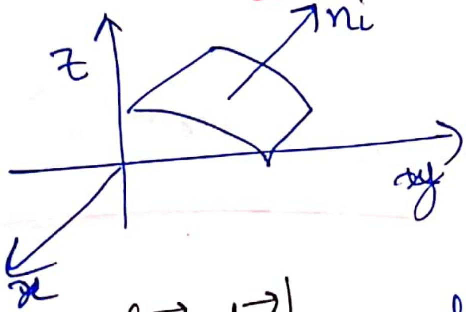
Def. $\boxed{n_i = \dfrac{\partial S}{\partial x^i}}$

i.e. $n_0 = \dfrac{\partial S}{\partial t}$

$n_1 = \dfrac{\partial S}{\partial x}$

$n_2 = \dfrac{\partial S}{\partial y}$

$d\vec{x} = dx^0 e_0 + dx^1 \ldots$



then why $dS \neq 0$

$g(\vec{n}, d\vec{x}) = n_i\, dx^i = \dfrac{\partial S}{\partial x^i} dx^i = dS$

as we travel dx^i is tangent we are moving

∴ $\sum_i n_i\, dx^i$ tells us the change in S

But $n_i\, dx^i$ solo would tell us change in S in dx^i direction

(22) Now let dx^i be tangent to surface ∴ we move along surface & surface is const.

then $dS = 0$ as $S = \text{const}$

∴ $n_i\, dx^i = 0 = g(\vec{n}, d\vec{x})$ ∴ $\vec{n}$ is $\perp^r$ Tangent

18.

$\therefore$ Seeing it like inner product

But in M it is not $\perp^r$.

$\therefore n_i \perp^r$ to $d\vec{x}^i \rightarrow n$ is not unit normal; 19.

$\therefore n$ is normal. $u^i v_i \rightarrow$ Null vector

(23) In Minkowski space if $g(u,u)=0$ then $v \not\perp v$ due to the negative signs in metric.

Eg. $q^i q_i = (q^0)^2 - (\vec{q})^2$

Let $q^i q_i = 0 \longrightarrow$ for Null vector.

$\therefore (q^0)^2 = (\vec{q})^2$

$g(\vec{P},\vec{P})=0$ Momentum of photon is such a vector.

$\therefore q^i \perp q_i$

$\boxed{\therefore \text{Define Normal to the Surface} \equiv n_i \equiv \frac{\partial S}{\partial x^i}}$

(24) Reimanian Spacetime signature has some -ve
Euc. spacetime. signature all +ve.

(25) $\therefore$ By (18) Normal transforms like covariant vector

(26) Gradient.

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Important.

$$\boxed{\begin{aligned} \partial_i\phi &= \left\{\frac{\partial\phi}{\partial x^i}\right\} \\ \partial^i\phi &= \eta^{ij}\partial_j\phi \\ &= \left\{\frac{\partial\phi}{\partial x_i}\right\} \end{aligned}}$$

$$\frac{\partial\phi}{\partial x^i} = \left(\frac{\partial\phi}{\partial x^0}, \vec{\nabla}\phi\right)$$

dx^i on the surface which is constant

$$\therefore n_i = \frac{\partial S}{\partial x^i} = \left(\frac{\partial S}{\partial x^0}, \vec{\nabla} S\right)$$

$\boxed{\vec{n}_i \text{ is Grad. of } S.}$

$$\boxed{\partial_i S \equiv \frac{\partial S}{\partial x^i}}$$

$\therefore \partial_i$ is the Gradient operator.

(27) Divergence

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\partial_i A^i \equiv \frac{\partial A^i}{\partial x^i} = \frac{\partial A^0}{\partial t} + \frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z}.$$

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$$\therefore \partial_i A^i = \frac{\partial A^0}{\partial t} + \vec{\nabla}\cdot\vec{A} \quad \text{--- (1)}$$

(28) **Continuity Eq^n** — $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$

Let $J^\alpha = (\rho, +\vec{j})$ $\boxed{\text{Current Density}}$

in (1) Put $A^i = J^i$

$$\therefore \partial_i A^i = \frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J}$$

$$= \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} \equiv \partial_i J^i$$

(29) $\partial_i A^i = 0$ (conservation law)
If Divergence is 0 then conservation law holds.

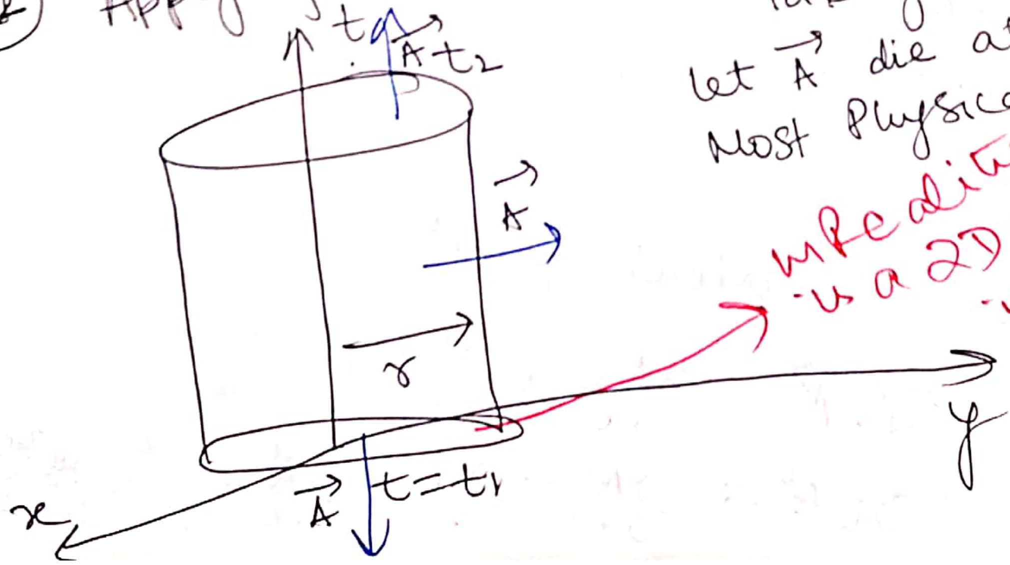
(30) Gauss Theorem

$$\boxed{\int_{3V} (\vec{\nabla}\cdot\vec{A})\, d^3x = \int_{\partial(3V)} (\vec{A}\cdot\vec{n})\, d^2\vec{x}}$$

$\partial(3V)$ ||| Boundary of volume.

$$\therefore \boxed{\int_{4V} (\partial_i A^i)\, d^4x = \int_{\partial(4V)} (A^i n_i)\, d^3\vec{x}}$$

(32) Applying the above theorem to cylinder



Taking $r = \infty$
let $\vec{A}$ die at ∞
Most Physical Qty die at ∞
in reality this is a 2D sphere
y & z is taken

$$\int_{4V} (\partial_i A^i)\, d^4\vec{x} = \int_{t=t_1}^{t_2} (A^i n_i)\, d^3\vec{x}$$

Curved part Integ. 0 as $\vec{A} = 0$ at ∞.

$\therefore$ Only upper & lower part contributes.

$$\therefore \int_{t=t_1}^{t=t_2} (A^i n_i)\, (dx\, dy\, dz)$$

By def. of normal $\boxed{n_i = \partial_i t}$ (as $t =$ const is there i.e. Surface)

$\boxed{\text{Diverg. } \cancel{\text{Gradient}} \text{ of Tensor } \partial_e A^{ijkem\cdots n}}$

$n_0 = 1 \cancel{+0+0+0}$

$n_0 = 1$ $\quad \boxed{n_1 = 0 \quad n_2 = 0}$

$\boxed{\dfrac{\partial t}{\partial x} = 0}$ as t & x are Ind. variables

$n = (1, 0, 0, 0) = (n_0, n_1, n_2, n_3)$

$A^i n_i = A^0 n_0 = A^0$

$$\therefore \int_{t=t_1}^{t=t_2} A^0\, dx\, dy\, dz = \int A^0\, dx\, dy\, dz \Big|_{t_2} - \int A^0\, dx\, dy\, dz \Big|_{t_1}$$

$-$ve sign signify outer pointing normal.

(32) Assuming $\partial_i A^i = 0$

$$\therefore \int_{t=t_1} A^0\, d^3x = \int_{t=t_2} A^0\, d^3x = \underline{\underline{\text{Const}}}.$$

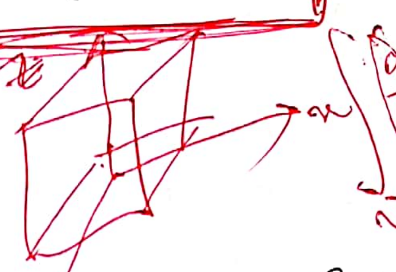
(33) Whenever there is $\partial_i A^i = 0$ You can write down Conserved Qty ($\because A^0$ is Conserved)

(34) The same proof holds for n Rank tensor $A^{ijk\cdots}$ if $\partial_e A^{ijke\cdots} = 0$ for any 'l'

$$\Rightarrow \int_{t=t_1} A^{0ijkom\cdots}\, d^3x = \text{Const}.$$

(35) Gauss theorem also holds for any f^n which is not vector.

(f^1, f^2, f^3)



$$\int \left(\frac{\partial f^1}{\partial x}+\frac{\partial f^2}{\partial y}+\frac{\partial f^3}{\partial z}\right) dx\,dy\,dz$$

$$= \int f^1\, dy\,dz + - + - = \int (\vec{A}\cdot\vec{n})\, d^2\vec{x}$$

$\int_V \frac{\partial f^1}{\partial x}\, dx\, dy\, dz$

$\int_{\partial V} f^1\, dy\, dz$

∴ Input can be a vector or not.

(36) Lowering of index for Tensors

$$M^{ij} \longrightarrow M^i{}_j = \eta_{jk} M^{ik}$$

(37) Contraction

$$M^i{}_j\, \delta^j{}_i = M^i{}_i \quad \text{(Trace of Matrix)}$$

If there are more than 2 indices then it is contraction.

It reduces the rank by 2.

(38) Epsilon Tensor

in 3D upper & lower index doesn't matter ∴ $\epsilon_{\alpha\beta\gamma} A^\beta B^\gamma$

Cross Product $(A\times B)^\alpha = \epsilon^{\alpha\beta\gamma} A_\beta B_\gamma$ where $\alpha,\beta,\gamma = 1,2,3$

$\epsilon^{\alpha\beta\gamma}$ is antisymmetric in all the indices i.e. if we interchange any 2 indices the sign flips.

& if 2 indices are same then it is 0

i.e. $(A\times B)^2 = \epsilon^{2\beta\gamma} A_\beta B_\gamma$

31, 13 } all others would lead to 0

we can't do this in 4D

i.e. $(A \times B)^{\alpha} \neq \epsilon^{\alpha\beta\gamma\eta} A_{\gamma} B_{\eta}$

Here two indices α, β are left

(39) in 3D

$A \times B$ 2 vectors together maps to 3rd

But we shouldn't think of $A \times B$ as vector

In same way,

Rotation is thought to be about an axis.

Rotation should be thought of as always in a plane.

in 3D. Rotation in XY plane $\equiv$ Rotation abt z axis

in 4D Rotation in x y plane $\equiv$ Is it Rotation about 3rd or 4th axis.

$\therefore$ Rotation in a plane def. carries out in d dimensions. But Rotation about axis is 3D specific BCZ $3-2=1$

in similar way Cross Product should be thought of as Antisymmetric Tensors.

(40) $A_{\alpha} B_{\beta} = 9$ components of Matrix in 3D

If the matrix is antisymmetric then Diag. will vanish

$\therefore$ 6 paired up left

$\therefore$ 3 components $\Rightarrow$ $\therefore$ we can map it another vector.

(41) $A_{\alpha} B_{\beta} = 16$ components in 4D

If the matrix is antisymm. then 4 Diag. elents $=0$

$\therefore$ 12 paired up.

$\therefore$ 6 components $\therefore$ Can't map to 4 comp.

ϵ^{ijkl} is useful to construct Duals.

(42) We can contract $\epsilon^{ijkl}\,\delta^{n}_{l}$ we get 3 Rank Tens.

$\epsilon^{ijkl}\,\delta^{nm}_{kl}$ we get 2 Rank Tens

$\epsilon^{ijkl}\,\delta^{nmo}_{jkl}$ we get vector

(43) Inner Product generalized to arbitrary space But cross Product doesn't.

(44) <u>Dynamics in CM</u>

$$A[q(t);\, q_1,t_1;\, q_2,t_2] = \int_{q_1,t_1}^{q_2,t_2} L(q,\dot{q},t)\,dt$$

A is fn of q_1,t_1 & q_2,t_2

A is functional of $q(t)$

We can think of A as fn of $q_1 t_1$ & $q_2 t_2$ (When Classical Action is considered)

or

A as functional of $q(t)$ with endpts fixed.

Around x_0

$f(x) = f(x_0) + f'(x)\big|_{x_0}(x-x_0) + f''(x)\big|_{x_0}\dfrac{(x-x_0)^2}{2!} + f'''(x)\big|_{x_0}\dfrac{(x-x_0)^3}{3!}$

(45)
$$\delta A = \int dt \left\{ \frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta\dot{q} \right\}$$

Around x_0

$f(x_0+\epsilon) = f(x_0) + f'(x_0)\,\epsilon + f''(x_0)\dfrac{\epsilon^2}{2!}$

$$\delta\dot{q} = \frac{d}{dt}\,\delta q$$

$$\delta A = \int dt \left\{ \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,\delta q\right) \right\}$$

$$= \int dt \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right\} + \frac{\partial L}{\partial \dot{q}}\,\delta q \Big|_{t=t_1}^{t=t_2}$$

... = 0 at Boundary, ... to make it ... $\Rightarrow$ Charge

This is the general Expression of Variation of Action.

If we fix end pts then $\delta q = 0$
& if we demand $\delta A = 0$
then EOM is obtained.

* $L(q, \dot{q}, t)$

$$L'(q, \dot{q}, t) = L + \frac{d}{dt} f(q, t)$$

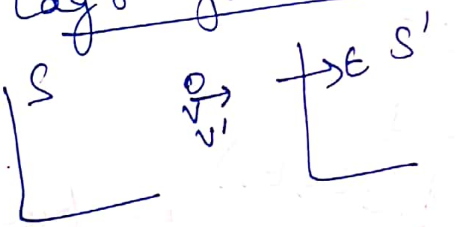
Both gives same EOM

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt = S_1 + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

$\hookrightarrow$ Action is not invariant
See last pt.

$$\therefore \delta S' = \delta S_1$$

* <u>Lagrangian for free particle</u>



(Galilean Transf^n)

$$v = v' + \epsilon$$

$$\therefore v' = v - \epsilon$$

But for free particle $L(v^2)$
& <u>By SR 1st Postulate</u> $L'(v'^2) = L(v^2) + \frac{d}{dt} f(x, t)$

EOM same
Remain Invariant.

or

$$L'(v'^2) = L(v^2)$$

$$L'(v'^2) = F(\epsilon)$$

Taylor Exp. around $\epsilon = 0$

$$F(\epsilon) = F(0) + \left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2)$$

$$\left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left(\frac{\partial F}{\partial v'^2}\right)\left(\frac{\partial v'^2}{\partial \epsilon}\right)\right|_{\epsilon=0}$$

$$= \left.\frac{\partial F}{\partial v'^2}(2\epsilon - 2v)\right|_{\epsilon=0}$$

$$= -\frac{\partial F}{\partial v'^2}(2v)$$

$$F(\epsilon) = F(0) - \left.\left(\frac{\partial F}{\partial v'^2}\right)\right|_{\epsilon=0}(2v\epsilon) + O(\epsilon^2)$$

* $F(0) = L(v^2)$

$\therefore \quad L'(v'^2) = L(v^2) - \frac{\partial L}{\partial v^2}(2v\epsilon)$ By 1st SR Post.

$\therefore \quad \frac{\partial L}{\partial v^2} = \text{constant}$

$\therefore \quad L \propto v^2$

$\Rightarrow L = \frac{m}{2}v^2$

as $L = \frac{mv^2}{2}$

$L' \neq \frac{mv'^2}{2}$ $\quad L' = \frac{mv^2}{2} + \frac{d}{dt}(f(x,t))$

* Lagrangians are not invariant under Galilean Transfn. Action is also Not invariant

$$S' = S + f(q^2, t_2) - f(q^1, t_1)$$

* Only EOM are invariant.

$\frac{mv'^2}{2}$ & $\frac{m\epsilon^2}{2}$ x $m(v, \epsilon)$

How?

(46) $A \longrightarrow A_c[q_c(t); q, t; q_1, t_1]$
↳ Classical Action

Now treating Action for fixed functional $q_c(t)$ from where EOM has been found & treat Action as fn of q_1, t_1 & q_2, t_2 & making $q_2, t_2 = q, t$ (variable). & fixing initial pts q_1, t_1.

see (17) L3

(47) ∴ $\boxed{\delta A_c = \frac{\partial L}{\partial \dot q}\delta q}$ as in (45) for EOM $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$ ∴ $\delta A_c = \frac{\partial L}{\partial \dot q}\delta q$

∴ $\boxed{\frac{\partial A_c}{\partial q} = \frac{\partial L}{\partial \dot q} = p \equiv \text{Canonical Momentum.}}$

If we know Action as fn of End pts we can find Canonical momentum without even knowing L. It is useful where Action is well defined But L is not.

(48) $\frac{\partial A}{\partial t} + H\left(\frac{\partial A}{\partial q}, q\right) = 0$ (Hamilton Jacobi)

$\frac{\partial A}{\partial \vec q} = \vec p, \quad H = -\frac{\partial A}{\partial t}$

$\int \mathcal{L}\, dt$ is special kind of functional in STR $\int \mathcal{L}\, d\tau$ → local lag. Density

(49) Lagrangian in CM

→ Variation of classical action w.r.t. end pt. variation $= p$.

Action is well defined But lag. is not.

where do we see it?

→ Actions is just functional of ~~coordinate~~ path Nobody told me it should be $\int L\, dt$ Integral over local function

L-3

① $A = \int dt\, L(q, \dot{q}, t)$

By Inertial frame def.

~~for~~ Free particle $L = L(v^2)$

Further ~~to~~ we have to induce 2 Principles.

② Galilean Relativity = Laws of Physics remain invariant under Galilean Transf^{n}

$$L' = L + \frac{d}{dt} f(x,t)$$

Using both these we get $L = \frac{m}{2} v^2$

③ Here we have $x^i(s)$: Trajectory of Particle.

→ Gives Position of Particle as fn of proper time

$\therefore A[x^i(s)]$ functional

④ Assumption $A[x^i(s)]$ is invariant under Lorentz Transf.

Convention

$\therefore \boxed{A[x^i(s)] = -k \int ds}$

for free Particle

In CM Action, Lag. were not Inv. under Galilean Transf.

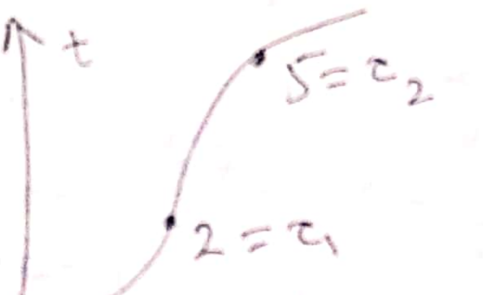
⑤ $A = -k \int \frac{dt}{\gamma}$

$\therefore \boxed{L = \frac{-k}{\gamma}}$

$A[x^i(s); t_1, x_1, y_1, z_1; t_2, x_2, y_2, \ldots]$
functional of $x^i(s)$
fn of x_1, t_1, x_2, t_2

$s = \tau$

⑥



$x^i(s) = x^i(\tau) \quad \forall\, i = 0, 1$

(7) Now

as $L_1 = -\frac{k}{\gamma} = -k(1-v^2)^{1/2}$

when $v << c$ or $v << 1$

$L = \frac{mv^2}{2}$ But from ours $L_1 = -k\left(1 - \frac{v^2}{2}\right)$

$\approx -k + \frac{kv^2}{2}$

Physical meaning.

$L_1 = -k + \frac{kv^2}{2}$

$L_1' = \frac{kv^2}{2}$

$L_1 = L_1' + \frac{d}{dt}(-kt)$

$\therefore L_1, L_1'$ give same EOM

(8) comparing L_1' with L

we get $k = m$.

$\therefore L_1 = -m(1-v^2)^{1/2}$

$\therefore \boxed{L_1 = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$

As Lag & Hamit have same dimension

(9) EOM can be derived similarly

$\frac{\partial L}{\partial \vec{x}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right)$

$\boxed{\frac{\partial L}{\partial \vec{v}} \text{ acts as covariant vector}}$

also by $\frac{\partial L}{\partial \dot{q}} = p = m\gamma v$

(10) For free Particle $\frac{\partial L}{\partial x} = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = 0 \Rightarrow \frac{d}{dt}\left(+\frac{m\cdot v}{2}(1-v^2)^{-1/2}\right) = 0$

$\frac{d}{dt}(\gamma m v) = 0 \quad \therefore \underline{P = \gamma m v}$

(11)

$$H = \vec{p}\cdot\vec{v} - L$$

$$= \gamma m v^2 + \frac{m}{\gamma}$$

$$= \gamma m\left(v^2 + \frac{1}{\gamma^2}\right) = \gamma m$$

$$\therefore \boxed{E = \gamma m}$$

$$\boxed{E = \gamma m c^2}$$

But $H(q, p)$

(12) $E = \gamma m$

$P = \gamma m v$

$$E = (p^2 + m^2)^{1/2} = (\gamma^2 m^2 v^2 + m^2)^{1/2} = m\gamma$$

$$\therefore E = (p^2 + m^2)^{1/2}$$

$$E \approx \frac{p^2}{2m} + m$$

(13) In Non Rel. Case
when velocity is 0 ~~in Newtonian & Relativity~~

$p = 0$

$E = mc^2$ in Rel. → This has to do with $L = -\cancel{m} + \frac{mv^2}{2}$. When const. is added to L then in H that const is subtract. Here H adds +m. ∴ $L = -m + \frac{mv^2}{2}$

$E = 0$ in Newtonian

This tells E has been elevated to mc^2

(14) ~~Here we are using Lorentz invariance~~
~~But in $L = L(v^2)$ a constant in which $L = L(v^2)$~~
~~There is fn except~~
~~remain invariant.~~

(14) In Gal. Trans. Lag., Action was not invariant
Here Under Lorentz Transf., Lag, Action Both are invariant.

(15) $A = -m \int_{t_1,x_1,y_1,z_1}^{t_2,x_2,y_2,z_2} ds = -m \int \sqrt{dx_\alpha dx^\alpha}$

$\delta A = -m\, \delta \int ds = -m \int \delta \sqrt{dx_\alpha dx^\alpha}$

$\delta A = -m \int \frac{1}{2} \frac{1}{\sqrt{dx_\alpha dx^\alpha}} \cdot 2\, dx_\alpha\, \delta(dx^\alpha)$

$= -m \int \frac{dx_\alpha}{ds}\, \delta(dx^\alpha)$

$= -m \int u_\alpha\, d(\delta x^\alpha)$

$= -m \int d(u_\alpha \delta x^\alpha) + m \int \left(\frac{du_\alpha}{ds} \delta x^\alpha \right) ds$

Action is invariant automatically makes lag. inv.

$\delta A = -m u_\alpha \delta x^\alpha \Big|_{t_1,x_1}^{t_2,x_2} + m \int ds \left(\frac{du_\alpha}{ds} \right) \delta x^\alpha$

Assuming End Points fixed $\delta x^\alpha = 0$ at 1, 2

$\therefore$ EOM

$\frac{du_\alpha}{ds} = 0 \Rightarrow \frac{du^\alpha}{ds} = 0 \quad \forall \alpha$.

~~When curve is~~ Affine parametrized

if not $\Rightarrow \frac{du_i}{d\lambda} = u_j \frac{d \ln L}{d\lambda}$

(16) As $\frac{du^\alpha}{ds} = 0$

General feature of SR

$u^\alpha = (\gamma, \gamma \vec{v})$

for spatial part

Compare with (10)

$\frac{d(\gamma \vec{v})}{ds} = \gamma \frac{d(\gamma \vec{v})}{(dt)} = 0 \Rightarrow \frac{d(\gamma \vec{v})}{dt} = 0$

~~for free Particle?~~

$\frac{du^0}{dt} = \frac{d\gamma}{dt} = 0$

Redundant.

~~where I have Used?~~

(17) Now for Classical Action.

$$\delta A_c = -m\, U_a\, \delta x^a \Big|_{t_1, x_1}^{t_1, x} + m \int ds \left(\frac{dU_a}{ds} \right) \delta x^a$$

As done in (46) in L-2 → initial pt. t_1, x_1, y_1, z_1 → fixed
& end pts t, x, y, z

$$\therefore \frac{\delta A_c}{\delta x^a} = -m U_a = -m(\gamma, -\gamma \vec{v})$$

$$= (-E, \vec{p}) = -p_i$$

As $\frac{\delta A_c}{\delta x^a}$ is 4 momentum
$-m U_i$ is 4 momentum

$$\therefore \frac{\partial A_c}{\partial t} + E = 0$$

$$\frac{\partial A_c}{\partial \vec{x}} = \vec{p} \rightarrow \text{compare with (L2) 46, 47}$$

↳ 3 momentum

(18) $p^i = m U^i$ Let $\vec{P}$ = 4 momentum = $(E, \vec{p})$

$$p^i p_i = m^2 = E^2 - |\vec{p}|^2 = \eta^{ab} p_a p_b$$

Compare with (L2)

$$\therefore \eta^{ab} \partial_a A_c \partial_b A_c = m^2$$ Rel. Hamilton Jacobi Equ^n

$$\eta^{ab} p_a p_b = m^2$$

(FQ)

(19) ∴ In 4 momentum E is the 0^{th} component comes from here.

(20) Acceleration

$$a = \frac{du^i}{ds}$$

in (15) $a^i = \frac{du^i}{ds} = 0 \;\forall i$

(21) Th. $g(\vec{a}, \vec{u}) = 0$

Proof. as $g(\vec{u}, \vec{u}) = u^i u_i = 1$

$$\frac{d(u^i u_i)}{ds} = \frac{d}{ds} g(\vec{u}, \vec{u}) = u_i \frac{du^i}{ds} + u^i \frac{du_i}{ds}$$

$$= \eta_{ij} u^j \frac{du^i}{ds} + \eta_{ij} \frac{du^j}{ds} u^i$$

$$= 2 \eta_{ii} u^i \frac{du^i}{ds}$$

See 1st page → Doubt? $= 2 u_i \frac{du^i}{ds}$

$$= 2 a^i u_i = 0$$

$\therefore a^i u_i = g(\vec{a}, \vec{u}) = 0$

Imp: $a_i a^i = -\mathcal{E}^2 = -$ve qty. As this scalar, valid in any frame.

(22) In MCRF

$a^i u_i = 0 \Rightarrow a^0 u^0 - \vec{a} \cdot \vec{u} = 0$?

$\vec{U}_{MCRF} = (\vec{e}_0, 0, 0, 0)$

$\therefore \vec{u} = 0$

$u^0 = 1$

$\therefore a^0 = 0 \Rightarrow \vec{a}_{MCRF} = (0, a^1, a^2, a^3)$

Why isn't $\vec{a}_{MCRF}$ is $(0,0,0,0)$?

(23) Imp $m \frac{du^i}{ds} = \partial^i \phi$ (ϕ is the potential.) (Let say, This is the gen. of 3 vector law Newton's) gen. of $\vec{F} = \frac{d\vec{p}}{dt} = -\frac{\partial U(q,t)}{\partial q}$

But this will not work?

(i) (24) What is the diff B/W $\partial^i \phi$ & $\partial_i \phi$.

(25) Why in (23) generalization of Newton 3 vector law cannot hold?

Ans. $m \frac{du^i}{ds} = \partial^i \phi \Rightarrow m \frac{du_i}{ds} = \partial_i \phi$

But $u^i \partial_i \phi = m\left(u^i \frac{du_i}{ds}\right) = m u^i a_i = 0$

$\therefore u^i \partial_i \phi = 0$

$\frac{dx^i}{ds} \frac{\partial \phi}{\partial x^i} = 0$

Potential has to remain constant along the traj of the Particle.

$\frac{\partial \phi}{\partial x^i} \frac{dx^i}{ds} = \frac{d\phi}{ds} = 0$

Directional Derivative.

But

Particles can move in any direction & the directional derivative of $\phi = 0$

$\therefore$ This cannot hold true

$\therefore m \frac{du^i}{ds} \neq \partial^i \phi$

As LHS is 2-I. $\therefore$ RHS should also be $\phi(x^i)$ but only const is valid here.

$F_i = -\frac{\partial U}{\partial x^i}$

forces have to be vel. dep

(26) Only a particular kind of forces follow

$u^i \partial_i \phi = 0$

for which the above eqn is valid

(27) This is the reason forces in SR has to be velocity dependent. As if $A = -mc^2 \int ds - \int \phi(x^i) dt$

Not Lorentz Invariant

(Q) In Newton forces are not vel. dep.

& we want Action to Invariant

$\therefore A = -mc^2 \int ds - \int c\, ds$

(28) In Newtonian mechanics

in closed system

$$L = T(q_A, \dot{q}_A) - U(q_A)$$

in open system

$$L = T(q_A, \dot{q}_A) + T(q_B, \dot{q}_B) - U(q_A, q_B)$$

Let

$\circ$ A $\qquad\qquad$ $\circ$ B

Both are Interacting & we to find lag of A.

∴ Let us know the traj. of B $\quad q_B(t), \dot{q}_B(t)$

$$L = T(q_A, \dot{q}_A) + T(q_B(t), \dot{q}_B(t)) - U(q_A, q_B(t))$$

$$= T(q_A, \dot{q}_A) - U(q_A, t) + T(t)$$

$$= T(q_A, \dot{q}_A) - U(q_A, t) + \frac{df(t)}{dt}$$

$$\therefore \quad L_A = T(q_A, \dot{q}) - U(q_A, t)$$

(29) ∴ To free particle lag we can add $U(q_A)$

$$L = T(q_A, \dot{q}_A) - U(q_A) = \frac{mv^2}{2} - U(\vec{x}_A)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{x}}$$

$$\frac{d}{dt}(m\vec{v}) = -\frac{\partial U}{\partial \vec{x}}$$

$$F = -\frac{\partial U}{\partial \vec{x}} \quad \therefore \quad \vec{F} = \frac{d}{dt}(m\vec{v})$$

→ $L = -\frac{mc^2}{r} - \frac{e}{\gamma}$ → from which force $(\vec{v})$.

(30) But in SR

We cannot add any f^n of coordinates and obtain a Force Eqn due to (25)

i.e.

Potential has to remain constant along the traj. of particle

This is the very strong condition.

→ why action should be L.I.

(31) Another way of saying this is that Lorentz invariance is a very strong condition

→ Lorentz invariance restricts for us the possible nature of interactions which can exist.

(32) Phase Space & Distribution function

Def: No. of particle in phase space cell of area $d^3x\, d^3p$ $\equiv dN = f(\vec{x}, \vec{p}, t)\, d^3\vec{x}\, d^3\vec{p}$

(33) $\theta(p^0)\, \delta_D(p^2 - m^2)\, d^4\vec{p}\, F(p^0, \vec{p})$

$d^4\vec{p}$ = volume element in 4 Dim. Momentum space

It becomes Non zero when $p^2 = m^2$: δ_D

But we know $E^2 = p^2 + m^2$

→ θ = Heaviside step fn

$$\theta[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

→ as $E = \pm (p^2 + m^2)^{1/2}$

To make sure Only the Energy is refered we use $\theta(p^0)$ fn.

(34) $$\delta_D((p^0)^2 - E_{\vec{p}}^2)\, dp^0\, d^3p = \delta_D(p^2 - m^2)$$

where $E_{\vec{p}} = + \sqrt{|\vec{p}|^2 + m^2}$

(35) $$\delta(f(x)) = \sum_{Root} \frac{1}{|f'(x_R)|} \delta(x - x_R)$$ Property of delta fn

(36) $$\delta_D((p^0)^2 - E_{\vec{p}}^2) = \frac{1}{2E_{\vec{p}}} \delta_D(p^0 - E_{\vec{p}})$$ fn of p^0

(37) $$\underbrace{\theta(p^0)\delta_D(p^2 - m^2)\, dp^0\, d^3\vec{p}}_{\text{Lorentz invariant}}\, F(p^0, \vec{p}) = \frac{F(\vec{p})\, d^3p}{2E_{\vec{p}}}$$

$\therefore \dfrac{d^3\vec{p}}{E_p}$ is L.I.

(30)

(38) $m u^0 d^3\vec{x} = m \frac{dt}{ds} d^3\vec{x} = \underbrace{\frac{m d^4\vec{x}}{ds}}_{\text{This is Lorentz invariant}}$

$\therefore$ $E_p d^3x$ is L.I.

(39) $\therefore$ $d^3\vec{x}\, d^3\vec{p}$ is L.I.

(31) $\therefore$ Phase volume has to be Lorentz inv.

(40) $\int \underbrace{\frac{d^3 p}{E_p} f(\vec{x}, \vec{p}, t)}_{\text{L.I.}} \underbrace{p^a}_{\text{L.I.}} = J^a(\underbrace{\vec{x}, t}_{p \text{ is integerated down}}) \equiv$ Mass Current

(32) $\int \frac{d^3 p}{E_p} f(\vec{x}, \vec{p}, t) p^a p^b = T^{ab}(t, \vec{x}) \equiv$ Energy mom. Tensor

Some information is lost here

When integration is done Some information is lost. Momentum dispersion info is lost.

T^{00} = Average Energy $\langle E \rangle$

(41) $\vec{v} = \frac{\vec{P} c^2}{E}$

(42) for photon from Action Principle how to deal?
But $m = 0$
But in Action any const can be multiplied & we get same EOM $\therefore$ At End put $m \to 0$ lt.

(33)

(43) $\vec{v} = 1$

if $A_i B^i = 0 \Rightarrow A^{i'} B_{i'} = 0$

if $A_i B^i = k_1 \Rightarrow A^{i'} B_{i'} = f(x^{i'})$

as $\partial_{i'} x^{i} = f(x^{i})$

<u>in SR</u>

if $A_i B^i = k_1 \Rightarrow A^{i'} B_{i'} = k_2$

$\underline{\underline{k_1 \neq k_2}}$.

Scanned with CamScanner

L-1,2,3

$c^2 d\tau^2 = dl^2$ (for Timelike)

$c^2 ds^2 = -dl^2$ (for spacelike)

For acc. observer (in flat Spacetime)

$$d\tau = \frac{dt}{\gamma}$$

$$ds = +i\frac{dt}{\gamma}$$

16. Rel. Hamilton Jacobi

$p^i p_i = m^2 = E^2 - |\vec{p}|^2$

$\eta^{ab} p_a p_b = m^2$

17. $a_i U^i = 0$

$a_i = (0, a_1, a_2, a_3)$ in MCRF

$a_i a^i = -ve$ in MCRF.

18. $m\frac{dU_i}{d\tau} = \partial_i \phi$ (can't work)

See L. Eqn as Elongation & Contraction

19. Forces in SR vel. dep.

$F = \frac{\partial L}{\partial q} - \frac{d}{d\tau}\frac{\partial L}{\partial q}$

$L = -\frac{mc^2}{\gamma} \Rightarrow \therefore F(\vec{v})$

$A^i A_i$ = scalar

$U^i U_i = 1$ constant for timelike curve

$U^i U_i = \frac{dx^i dx_i}{ds^2} = -1$ const. for spacelike curve

20. $E^2 = p^2c^2 + m^2c^4$

for massless $E = pc$

(M, η_{ab}) Manifold in SR with metric

A^j, A_j are Equivalent.

$\vec{p} = \frac{E\vec{v}}{c^2}$

21. $E = m\gamma$, $p = m\gamma v$

$P_i = (E, \vec{p})$

Continuity eqn $\Rightarrow$ $\partial_i A^i = 0$

Gauss Theorem $\Rightarrow$ 4D $\int_v (\partial_i A^i)\, d^4x = \int_{\partial v} A^i d\sigma_i = \int_{\partial v} A^i n_i d\vec{x}^3$ if

3D

Conservation law if $\partial_i A^i = 0$ continuity eqn valid.

$(A \times B)_i = \epsilon_{ijk} A^j B^k$ in 3D

$= \epsilon_{ijk} C^{jk} \to$ Tot. A.S.

$= (C_i^*) \to$ Dual of C^{jk}

(9) Jacobian $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$J_{ij} (m \times n) = \frac{\partial f_j}{\partial x^i}$$

$$dV = J(U_1 U_2 U_3)\, dU_1 dU_2 dU_3$$ → Action, lag. not invariant under Galilean Transfn

CM

(10) EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

Classical Action

$$\frac{\delta A_c}{\delta q} = \frac{\partial L}{\partial \dot{q}} = p$$ ↳ conjugate momentum

(11) Conservation of Energy $\frac{dH}{dt} = 0$ for closed system when EOM follow

(12) Hamilton Jacobi Eqn

$$\frac{\partial A_c}{\partial t} + H(p,q) = 0$$

$$H(p,q) = \sum_i p_i \dot{q}_i - L$$

SR

(13) Action invariant under Lorentz. Transfn.

$$A = -k \int dl$$

$$k = mc \Rightarrow L = -\frac{mc^2}{\gamma}$$

(14) for CM & SR $L(v^2)$ free particle.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \Rightarrow \frac{d}{dt}(\gamma m v) = 0$$

$$\Updownarrow$$

(15) By least principle of Action $\frac{dU_i}{d\tau} = 0$

Time Component Redundant.

$$\frac{\delta A_c}{\delta x^i} = -p_i \Rightarrow \frac{\delta A_c}{\delta t} + E = 0$$

$$\frac{\delta A_c}{\delta x^i} = p_i$$

L-4

For free Particle — physical meaning: proper time being extremitized. → see Ch-1 Eric

① $A_R = -m\int ds$

$A_{NR} = \int \frac{mv^2}{2} dt$ → What sign of m s.t. H has a lower Bond? i.e. E should not go to $(-\infty)$. But for free Particle $H = \frac{mv^2}{2}$ ∴ Postulate H has a lower Bond

For Interaction

$A_{NR} = \int \frac{mv^2}{2} dt - \int V dt$ where $V(r,t)$

② for free Particle

$A_R = -m\int ds$

m positive in Relativistic case also

Proof: As A_R has to go to A_{NR} in NR limit ∴ m has to be +ve coz m is +ve in A_{NR}.

- As for free particle we wanted Action to Lorentz invariant → Assumption
- Int. particle also we want Action to be invariant

For Interaction — ϕ scalar

$A_R = -m\int ds - \lambda\int \phi\, ds$ → similar to $\int V dt$. This whole thing is Lorentz invariant ∴ ϕ is assumed to L.I.

λ: coupling const. ≡ Measures the strength the field ϕ couples to the particle.

ϕ: scalar field $\phi'(x') = \phi(x)$.

Comparing with ② we get $dt = ds$, $\phi = V$

4

(6) We could also have written

$$A = -m\int ds - \lambda\int \phi\, ds - \underbrace{\overset{\text{const.}}{q}\int A_j\, dx^j}$$

Action has integral over some world line.

(7) Another possibility.

$$A = -m\int ds - \lambda\int \phi\, ds - q\int A_j\, dx^j - k\int\sqrt{A_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\, ds$$

$$-\, l\int\sqrt[3]{A_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}}\, ds - - -$$

$$\int\sqrt{A_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\, ds = \int\sqrt{A_{ij}\, dx^i dx^j}$$

$\therefore$

$$A = -m\int ds - \lambda\int\phi\, ds - q\int A_j\, dx^j - k\int\sqrt{A_{ij}\, dx^i dx^j}$$

$$-\, l\int\sqrt[3]{A_{ijk}\, dx^i dx^j dx^k}$$

$$- - - -$$

(8) We could have also used $\int (d^n s)^n$ $\quad n = 2, 3 - - -$

We could have made it work By dividing By ds^2 & multiplying by ds.

e.g. $\int (d^3 s)^3 = \int \frac{(d^3 s)^3}{(d^2 s)^2}\, ds$

(9) $\int\sqrt{A_{ij}\, dx^i dx^j}$ Taking $A_{ij} = \eta_{ij}\phi^2$

then $\int\sqrt{A_{ij}\, dx^i dx^j} = \int\phi\, ds$ which is Equal to 2nd term

∴ This is more general & incorporates that we can ignore $\lambda \int \phi\, ds$.
∴ As this is the special case of $K\int\sqrt{A_{ij}\,dx^i dx^j}$.

11) ∴ we are left with vector, Tensor

$$A = -m\int ds \;-\; \underbrace{q\int A_j\, dx^j}_{\text{Particle is coupling with Vector field}} \;-\; K\underbrace{\int\sqrt{A_{ij}\,dx^i dx^j}}_{\text{Particle is coupling with Tensor field.}}$$

∴ Particle is coupling with Vector field — Electromagnetism

Particle is coupling with Tensor field. — Gravitation.

Why not take Extra 3 Rank Tensor?

12) As we know
$$ds^2 = \eta_{ij}\, dx^i dx^j = S^{ij} = \text{Symmetric Tensor}$$
& $A_{ij} S^{ij} = 0$

∴ we can choose 2nd Rank Tensor field to be Symmetric
as Antisym. part would go to 0.

13) **Project**

What happens if 3 Rank Tensor field is included?

14) For Current Purposes.

$$\boxed{A = -m\int ds - q\int A_j\, dx^j}$$

m, q : are properties of particle.

(15) $\boxed{A^j = (\phi, \vec{A})}$

Assumption A^j is a contravariant vector exists

ϕ will become Electrostatic Potential
$\vec{A}$ will become Vector Potential.
But we have not shown this yet.

(16) $A_j = (\phi, -\vec{A})$

$A_j dx^j = \phi\, dt - \vec{A}\cdot d\vec{x}$

$= \phi\, dt - \vec{A}\cdot \frac{d\vec{x}}{dt}\, dt$

$\therefore A = -m\int \frac{dt}{\gamma} - q\int \phi\, dt + q\int (\vec{A}\cdot\vec{v})\, dt$

$\therefore \boxed{L = -\frac{m}{\gamma} - q\phi + q(\vec{A}\cdot\vec{v})}$

(17) Here Lagrangian has velocity dependence
in NR $\quad L = \frac{mv^2}{2} - U(\vec{r_i})$

$\therefore \vec{F}$ are not velocity dep.

But Now in Relativity forces are velocity dep.
which is Consistent with (25) L-3.

(18) $\delta A = -m\,\delta\int ds - q\,\delta\int A_j dx^j$ on the path of particle

$\delta A = -m\int \underbrace{\frac{dx_\alpha}{ds}\,\delta(dx^\alpha)}_{\text{As in (15) L-3}} - q\left[\int \partial_i A_j (\delta x^i)\, dx^j + \int A_j\, \delta(dx^j)\right]$

A_j is varying coz A_j is evaluated on the path of the Particle. & by δA we mean we are changing path of particle $\therefore$

$$\delta A = -mU_i\delta x^i\Big|_1^2 + m\int ds\left(\frac{dU_\alpha}{ds}\right)\delta x^\alpha - q\int(\partial_i A_j)(\delta x^i)\frac{dx^j}{ds}ds$$

$$+$$

$$\int d(A_j\delta x^j) - \int \delta x^j(\partial_i A_j)\frac{dx^i}{ds}ds$$

Trick is same to get ds at end as we had done in

Interchange i & j

$$\delta A = -mU_i\delta x^i\Big|_1^2 + m\int ds\left(\frac{dU_\alpha}{ds}\right)\delta x^\alpha - q\left[\int(\partial_i A_j)(\delta x^i)U^j\, ds\right] - qA_j\delta x^j\Big|_1^2 + q\int(\partial_i A_j)U^i ds\,\delta x^j$$

$$= -(mU_i + qA_i)\delta x^i\Big|_1^2 + m\int\left(\frac{dU_\alpha}{ds}\right)ds\,\delta x^\alpha$$

Replace with α, j with α

$$+ q\left[\int(\partial_i A_j - \partial_j A_i)(\delta x^j)U^i ds\right]$$

to make all δx^α

$$\delta A = -(mU_\alpha + qA_\alpha)\delta x^\alpha\Big|_1^2 + m\int\left(\frac{dU_\alpha}{ds}\right)ds\,\delta x^\alpha$$

$$+ q\left[\int(\partial_i A_\alpha - \partial_\alpha A_i)(\delta x^\alpha)U^i ds\right]$$

(22) Assuming with $\delta x^i|_1^2 = 0$

$\delta A = 0$

$$m\left(\frac{du_\alpha}{ds}\right) + q\left(\partial_i A_\alpha - \partial_\alpha A_i\right) u^i = 0$$

~~Def: Fi~~

Rewriting

$$m\left(\frac{du_i}{ds}\right) + q\left(\partial_j A_i - \partial_i A_j\right) u^j = 0$$

$$\boxed{\text{Def: } F_{ji} \equiv \partial_j A_i - \partial_i A_j}$$

(23) We can work out F_{ji} from A_i

F_{ji} is antisymmetric

$\therefore$ $\underline{F_{ji} = -F_{ij}}$ $\Rightarrow$ Diag. Terms Zero

(24) $\therefore$ $$m\frac{du_i}{ds} + qF_{ji}u^j = 0$$

$$m\frac{du_i}{ds} = qF_{ij}u^j$$

$$\boxed{p^k = \eta^{ki} p_i}$$

But $m u_i = p_i$

$\therefore$ $$\frac{dp_i}{ds} = qF_{ij}u^j \Rightarrow \frac{d(\eta^{ki} p_i)}{ds} = q\eta^{k}$$

$$\boxed{\frac{dp^i}{ds} = q F^i_j u^j}$$

This is the Lorentz force equation.

2) As in NR mechanics

$L = \frac{mv^2}{2} + U(\vec{r})$

One has to give $U(\vec{r})$ maybe Harmonic Potential or 6th to calculate

In same way

In SR

$L = -m\int ds - q\int A_j dx^j$

One has to give $A_j = (\phi, -\vec{A})$.

$\boxed{\begin{matrix} COG = 0 \\ DOC = 0 \end{matrix}}$

) Now

$$\frac{dp^i}{ds} = q F^i_j u^j$$

The force is dependent on u^j & F^i_j
$u^j \downarrow$ 4 velocity

1st Maxwell eqn is obtains $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

∴

7) $F_{01} = \frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x} = -\frac{\partial A^1}{\partial t} - \frac{\partial \phi}{\partial x} = \left[-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right]_1$

$F_{02} = \frac{\partial A_2}{\partial t} - \frac{\partial A_0}{\partial y} = -\frac{\partial A^2}{\partial t} - \frac{\partial \phi}{\partial y} = \left[-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right]_2$

$F_{03} = \frac{\partial A_3}{\partial t} - \frac{\partial A_0}{\partial z} = -\frac{\partial A^3}{\partial t} - \frac{\partial \phi}{\partial z}$.

$$\therefore \boxed{-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi = F_{01}\vec{e_0} + F_{02}\vec{e_1} + F_{03}\vec{e_3} = \vec{E}}$$

2nd Maxwell eqn is obtained
$\vec{\nabla} \cdot \vec{B} = 0$

(28) $\boxed{\vec{E} = (F_{01}, F_{02}, F_{03})}$

(29) $F_{12} = -\frac{\partial A^2}{\partial x} + \frac{\partial A^1}{\partial y} = -\frac{\partial A^y}{\partial x} + \frac{\partial A^x}{\partial y} = -B^z$ $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

$F_{21} = -F_{12} = B^z$

$F_{32} = -B^x$ $\boxed{F^{ij} = \epsilon^{ijk} B_k}$

$F_{13} = -B^y$

$$F_{ik} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & +B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{bmatrix}$$

$\vec{B} = (F_{32}, F_{13}, F_{21})$

(30) F_{ij} is 16 element Anti symmetric matrix

As it is Ant. sym. $\therefore F_{ii} = 0 \quad \forall i$

Remaining (12)

$\frac{1}{2}$ are negative of other $\frac{1}{2}$

$\therefore$ 6 components

$3(\vec{B})$ & $3(\vec{E})$

(31) 4 Momentum $(E, \vec{P})$

4 Vector Potential $(\phi, \vec{A})$

But $\vec{E}$ & $\vec{B}$ don't come like this

i.e. adding time comp. & making them 4-vector

Though they come from comp. of Tensorial object F_{ij} (AS 2nd Rank Tensor)

(32) $$\frac{dP_i}{ds} = q\, F_{ij} \frac{dx^j}{ds}$$

writting $ds = \frac{dt}{\gamma}$.

$$\frac{dP_i}{dt} = q\, F_{ij} \frac{dx^j}{dt}$$

(33) $$F_{ij}\frac{dx^j}{dt} = F_{10} + \cancel{F_{11}\frac{dx}{dt}} + F_{12}\frac{dy}{dt} + F_{13}\frac{dz}{dt}$$

But for Ant Sym Tens. $F_{ii} = 0 \quad \forall i$

from (27) $F_{10} = -E^1$

$F_{12} = -B^z$

$F_{13} = B^y$

$$\therefore F_{ij}\frac{dx^j}{dt} = -E^1 - B^z v_y + B^y v_z$$

$$= -E^1 - (\vec{V}\times\vec{B})^1$$

$$\therefore \frac{dP_1}{dt} = -q\left(E^1 + (\vec{V}\times\vec{B})^1\right)$$

$$\Rightarrow \frac{dP^1}{dt} = q\left(E^1 + (\vec{V}\times\vec{B})^1\right)$$

$v << c$
$\vec{P} = m\vec{V}$
Eqn $\rightarrow$
$\therefore$ Lorentz force Eqn

$$\therefore \boxed{\frac{d\vec{P}}{dt} = q\left(\vec{E} + (\vec{V}\times\vec{B})\right)}$$ Relativistic Lorentz Force

(34) Once I know $\vec{E}$ & $\vec{B}$ I know $\vec{v}(t)$
By solving this

$\therefore$ I know $P^0 = m\gamma$ changes with time i.e. $\frac{dP^0}{dt}$

$\therefore \frac{dP^0}{dt}$ is not ind. of $\frac{d\vec{P}}{dt}$ $\rightarrow$ L-3 (16)

35) This is the genral feature in SR due to $a^i u_i = 0$ L-3 16

36) Find $\frac{dE}{dt}$? = work done By force

$\rightarrow$ in Free theory
$P_i \equiv m v_i$
$\frac{\partial L}{\partial \dot{q}} = \vec{p}$ $\quad H = p\dot{q} - L$

$\rightarrow \frac{\delta A_c}{\delta x^i} \equiv -P_i$ (in Interacting Theory)

$\frac{\partial L}{\partial \dot{q}} \equiv \vec{p}$ $\quad H = p\dot{q} - L$

37) Th. $a^i u_i = 0$ $\quad \frac{dP_i}{dt} = q F_{ij} u^j \Rightarrow m \frac{du_i}{dt} = q F_{ij} u^j$

Proof: $\underset{\uparrow \text{ Antisym}}{F_{ij}} \underbrace{u^j u^i}_{\text{Sym}} = A_{ij} S^{ij} = 0$

$a^i u_i = 0$ (from Def)
But we have shown this in particular case.

8) Classical Action

$\delta A_c = -(m u_i + q A_i) \delta x^i$

$\boxed{\frac{\delta A_c}{\delta x^i} = -(m u_i + q A_i) = -P_i}$ $\quad$ Compare 17 L-3

(39) Canonical ~~mo~~ momenta which should be ~~Momentum of Particle~~ the mom. of Particle is picking up the term "which depends on field".

Firstly $p_i = m u_i$

Now $P_i = m u_i + q A_i$

(40) from (16)

$$\frac{\delta A_c}{\partial t} + E + q\phi = 0$$

$$L = -\frac{m}{\gamma} - q\phi + q\,\vec{A}\cdot\vec{v}.$$

But as $\vec{P} = \frac{\partial L}{\partial \vec{v}} = m v \gamma + q\vec{A}$

Canonical mom. picks up the field dep. term.

which is consistent with (38)

$$\frac{\delta A_c}{\partial t} + (E + q\phi) = 0$$

$$\frac{\delta A_c}{\delta \vec{x}} = \vec{p} \cancel{+ q\vec{A}}$$

(41) Compare with (18) L-3

$$(P_i - q A_i) = m u_i$$

$$m^2 u_i u_j \eta^{ij} = m^2$$

$$(P_i - q A_i)(P_j - q A_j)\eta^{ij} = m^2$$

$$\boxed{\left(-\frac{\partial A_c}{\partial x^i} - q A_i\right)\left(-\frac{\partial A_c}{\partial x^j} - q A_j\right)\eta^{ij} = m^2}$$

Rel. Hamilton Jacobi Eqn.

free Particle Rel. H-J Eq. $\boxed{\frac{\partial A_c}{\partial x^i}\frac{\partial A_c}{\partial x^j}\eta^{ij} = m^2}$

(42) $F_{ij} = \partial_i A_j - \partial_j A_i$

$$\frac{dp^k}{ds} = q F^k_j u^j$$

Let particles be moving in EM field then particles acc. would depend only on F^k_j ∴ we will get F^k_j (6 comp) i.e. $\vec{E}, \vec{B}$.

But we want to know A_j given F_{ij}?

(43) $A_j \rightarrow \boxed{\bar{A}_j \equiv A_j + \partial_j f}$

$\bar{F}_{ij} = F_{ij} + \cancel{\partial_i \partial_j f} - \cancel{\partial_j \partial_i f}$

$\therefore \bar{F}_{ij} = F_{ij}$

$\bar{\Phi} = \phi + \frac{\partial f}{\partial t}$
$\vec{A}' = \vec{A} - \vec{\nabla} f$

Gauge Transfn.

$\vec{B}' = \vec{B}$
$\vec{E}' = \vec{E}$

Bcz F_{ij} is invariant & A_j can be not uniquely determined

$\boxed{\therefore \text{Given } F_{ij} \text{ we Cannot Obtain uniquely } A_j.}$

(44) A_j is the field we added to Lag. But we cannot measure it. as $L = -\frac{m}{\gamma} - q\phi + q\vec{A}\cdot\vec{v}$

(45) Th. ~~Action~~ EOM is invariant under Gauge Transfn.

$-q \int A_k dx^k \longrightarrow -q \int A_k dx^k - q \int (\partial_k f) dx^k$

$= \text{Original} - f \Big|_1^2$

This will not disturb EOM

NO 4 TREE
$\therefore$ Charge is conserved automatically in this case as q is here parameter. But GaugeT is req to make charge cons.?

(46) $\therefore$ The other way around if we know ~~Action~~ EOM is inv. under Gauge Trans $\therefore F_{ij} = F_{i'j'}$
$\therefore$ ~~EOM would Remain Same~~
$\therefore$ In EOM if we do G.T. they would remain same in (43).

(47) m +ve due to H bounded
q +ve, −ve with the same argument.

(48) q is invariant? q is just the parameter in the action.

L-S

→ As we know $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$

$\vec{B} = \vec{\nabla} \times \vec{A}$

By $\vec{A} \to \vec{A} - \vec{\nabla} f$

$B = \vec{\nabla} \times (A - \vec{\nabla} f) = \vec{\nabla} \times \vec{A}$ Remains Inv.

But $\vec{E} = -\frac{\partial A}{\partial t} + \frac{\partial \vec{\nabla} f}{\partial t} - \vec{\nabla}\phi$ changes

E Remains Inv. when $\phi \to \phi + \frac{\partial f}{\partial t}$

(1) As $\vec{E}, \vec{B}$ are components of F_{ij} & we know $F_{i'j'} = L^k_{i'} L^l_{j'} F_{kl}$ how $F_{i'j'}$ transform under L.T.

$\therefore$ we know how $\vec{E}$ & $\vec{B}$ transform under L.T.

(2) $\vec{E}'_{\parallel} = \vec{E}_{\parallel}$ $\qquad$ $\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + (\vec{v} \times \vec{B})_{\perp})$

$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$ $\qquad$ $\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - (\vec{v} \times \vec{E})_{\perp})$

Sol^n ⇒

$F_{0'1'} = E'_{\parallel} = L^i_{0'} L^j_{1'} F_{ij}$
$= L^i_{0'}(\gamma v F_{i0} + \gamma F_{i1})$
$= \gamma^2 v^2 F_{10} + \gamma^2 F_{01}$
$= -\gamma^2 v^2 E_{\parallel} + \gamma^2 E_{\parallel}$
$= \gamma^2 E_{\parallel}(1 - v^2)$
$= E_{\parallel} = F_{01}$

→ $F_{0'2'} = E'_y = L^i_{0'} L^j_{2'} F_{ij} = L^i_{0'}(F_{i2})$
$= \gamma F_{02} + \gamma v F_{12} = \gamma(E_y) + \gamma v(-B_z)$
$= \gamma[E_y + (\vec{v} \times \vec{B})_y]$

→ $F_{2'1'} = B_z' = L^i_{2'} L^j_{1'} F_{ij} = L^i_{2'}(\gamma v F_{i0} + \gamma F_{i1})$
$= \gamma v F_{20} + \gamma F_{21}$
$= \gamma B_z + \gamma v(-E_y) = \gamma[B_z - (\vec{v} \times \vec{E})_z]$

→ $F_{3'2'} = B'_{\parallel} = L^i_{3'} L^j_{2'} F_{ij}$
$= L^i_{3'}(F_{i2}) = F_{32}$
$= B_{\parallel}$

(3) $\therefore$ $\vec{E}$ & $\vec{B}$ are Not Lorentz invariant qty.

(4) We want to construct Lorentz invariant qty from F_{ab}.

(5) Def $\vec{F} \equiv \vec{E} + i\vec{B}$

$F'_y = \gamma[(E_y + (v \times B)_y) + i(B_y - (v \times E)_y)]$

$F'_y = \gamma[F_y + (v \times B)_y - (v \times E)^i_y]$

$x' = \gamma(x - vt)$

$\therefore$ F rotates in $(t-x)$ plane

$F'_x = F_x$ as $E'_{\parallel} = E_{\parallel}$ & $B'_{\parallel} = B_{\parallel}$

?

under L.T. in $(t-x)$ plane (F_y, F_z) gets Rotated → Put E_y, B_y & work out

?

$\therefore$ The coordinate is Rotating & the length remains invariant.

i.e. $g(\vec{F}, \vec{F})$ is invariant.

(6) $g(\vec{F}, \vec{F}) = g(\vec{E} + i\vec{B}, \vec{E} + i\vec{B})$

$$= g(\vec{E}, \vec{E}) + 2g(\vec{E}, i\vec{B}) - g(\vec{B}, \vec{B})$$

$$= g(\vec{E}, \vec{E}) - g(\vec{B}, \vec{B}) + 2i\, g(\vec{E}, \vec{B})$$

$$= E^2 - B^2 + 2i\, \vec{E}.\vec{B}$$

(7) Invariance of a Complex vector requires invariance of Both Img. & Real part.

$$\therefore \left.\begin{array}{l} E^2 - B^2 \\ \vec{E}\cdot\vec{B} \end{array}\right\} \text{Both are invariant.}$$

work this out.

$\vec{F} = \vec{E} + i\vec{B}$ L.T. ~~in x direction~~ along (y–z) plane

$F_y = E_y + iB_y$

$$\bar{F}_y = \bar{E}_y + i\bar{B}_y$$

By (2)

$$= \gamma(E_y + (\vec{v}\times\vec{B})_y) + i\gamma(B_y - (\vec{v}\times\vec{E})_y)$$

$$= \gamma(E_y + iB_y) + \gamma((\vec{v}\times\vec{B})_y - i(\vec{v}\times\vec{E})_y)$$

$$= \gamma[F_y + (\vec{v}\times\vec{B})_y - i(\vec{v}\times\vec{E})_y]$$

$$= \gamma[F_y + (\vec{v}\times(\vec{B} - i\vec{E}))_y]$$

$t' = \gamma(t - vx)$

$$\bar{F}_z = \gamma(F_z + (\vec{v}\times\vec{B})_z - i(\vec{v}\times\vec{E})_z]$$

$$= \gamma(F_z + (\vec{v}\times(\vec{B} - i\vec{E}))_z]$$

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(8) $F_{ab} F^{ab} = 2(\vec{B}^2 - E^2)$ → work out?

~~as~~ $g(\vec{A}, \vec{B})$ is invariant.

$$A^i B_i = A^{i'} B_{i'} = L^{i'}_j L^k_{i'} A^j B_k$$
$$= \delta^k_j A^j B_k$$

Similarly

$$F^{ab} F_{ab} = F^{a'b'} F_{a'b'} = L^{a'}_a L^{b'}_b L^c_{a'} L^d_{b'} F^{ab} F_{cd}$$
$$= \delta^c_a \delta^d_b F^{ab} F_{cd} = F^{ab} F_{ab}$$

(9) $\epsilon^{abcd} F_{ab} F_{cd} \propto \vec{E}.\vec{B}$ → work out.

(10) These are the only 2 possibilities to get something invariant which is $f(\vec{E}, \vec{B})$.

(11) Th. There can't be another invariant which is quadratic in $\vec{E}$ & $\vec{B}$.

∴ Coordinates is being Rotated But Norm of $\vec{F}$ vector is invariant.

Both Transfn are like 4vectors L. Transfn & as 4vector Transf is like Complex Rotation ∴ These are like Rotation ⇒ F_y & F_z are Rotating under L.T.

in 4 vect L.T. we can ~~also~~ see that in Diff. frames 4vectors are Rotated ⇒ we can also see that a 4vector Remains same & coordinate frame gets Rotated

(12) $$\left[\begin{array}{l}\vec{B} = \vec{\nabla} \times \vec{A} \\ \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\end{array}\right.$$

from F_{ij}:

& from $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

(6) $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(13) $$\boxed{\begin{array}{l}\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \\ \vec{\nabla} \times \vec{E} = -(\vec{\nabla} \times \vec{\nabla}\phi) - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = -\frac{\partial \vec{B}}{\partial t}\end{array}} \quad \text{— (1)}$$

Maxwell eqn should be seperated out for those which have source on RHS & those which do not have.

— (1) is completely independent of the charge & current living in the space.

The other two eqn depend on charges & current

(14) ∴ Once A_j has been introduced then 2 Maxwell eqn becomes vaquous.

55.

(15) $\partial_a(\epsilon^{abcd}F_{cd}) = \partial_a(\epsilon^{abij}\partial_i A_j + \epsilon^{abji}\partial_j A_i)$

$= \partial_a(\epsilon^{abcd}\partial_c A_d + \epsilon^{abcd}\partial_c A_d)$

Bianchi Identity

$= 2\partial_a(\epsilon^{abcd}\partial_c A_d)$

(16) Th. $\partial_a(\epsilon^{abcd}\partial_c A_d) = 0$

$\underbrace{\epsilon^{abcd}}_{\text{Antisy } a \& c}\ \underbrace{\partial_a\partial_c(A_d)}_{\text{Sym in } a \& c}$

$\therefore\ \partial_a(\epsilon^{abcd}\partial_c A_d) = 0 \Rightarrow \boxed{\partial_a(\epsilon^{abcd}F_{cd}) = 0}$

(17) Obtain (1) from $\partial_a(\epsilon^{abcd}\partial_c A_d) = 0$;

or

$\partial_a(\epsilon^{abcd}F_{cd}) = 0$

$\therefore$ We can Obtain maxwell's one set of eqn from F

$\therefore$ One A_j has been given it can construct F & hence from $\partial_a(\epsilon^{abcd}F_{cd}) = 0$ I can obtain Maxwell Eqn.

(1) $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

$(\vec{A}\times\vec{B})^i = \epsilon^{ijk}A_jB_k$

(2) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0$

$\partial_i(\epsilon^{ijk}\partial_j A_k) = 0$

$(2\epsilon^{ijk}\partial_i\partial_j A_k) = 0$

$\partial_i(*F)^{ij} = 0$

But in 3D A_k

$i=1,2,3$

(1) $F^{ij} = \epsilon^{ijk}B_k$

$i=1,2,3$ $F^{0i} = E^i$

(2) From Def. of F

$\partial_i(*F)^{ij} = 0$

$(*F)^{ij} = \epsilon^{ijkl}\partial_k A_l$

? in 4 dim $ijkl$

(12) (18) Dual of $F \equiv (F^*)$

$$(F^*)^{ab} \equiv \epsilon^{abcd} F_{cd}$$

$\therefore$ Maxwell's First two eqn-

$$\boxed{\partial_a (F^*)^{ab} = 0}$$

(19) for only one charge

$$A = -m \int ds - q \int A_j \, dx^j$$

(20) if we have more than 1 charge

$$- \sum_i q_i \int (A_j)_i \, dx_i^j$$

As in (18) & (4)

A_j has to be calculated on the traj of the charge particle.

Eg: for 2 charges

$$-q_1 \int A_j \, dx^j - q_2 \int A'_j \, dx'^j$$

where A_j is for charge q_1

A'_j is for charge q_2

(21) Charge Density

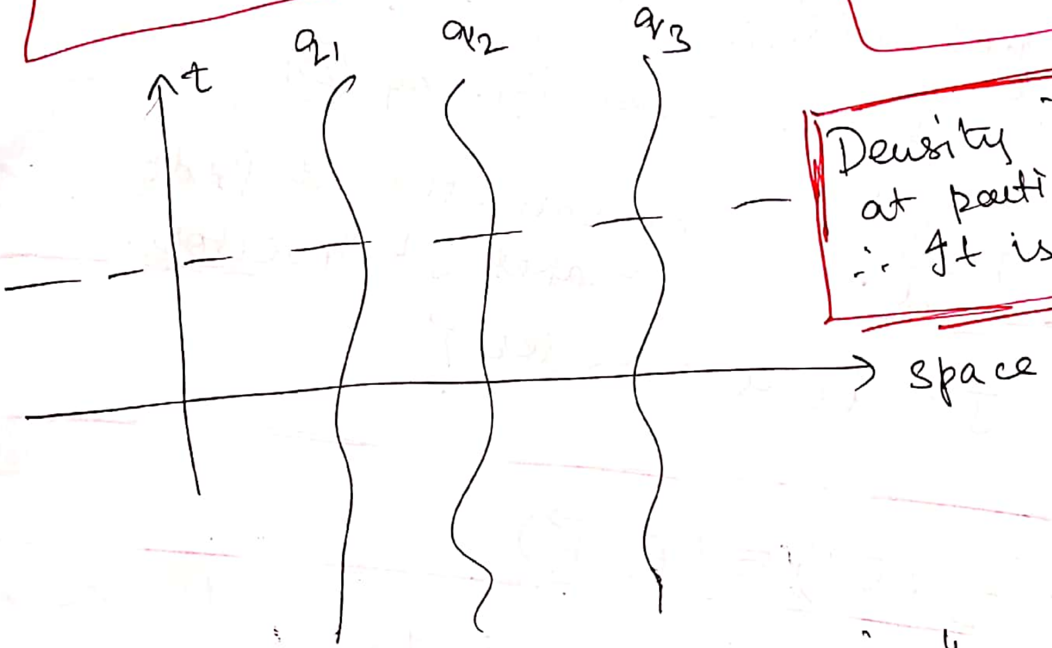
$$- \int A_j \rho \, dx^j \, d^3\vec{x} = -q \int A_j \, dx^j$$

$$\rho(t, \vec{x}) = \sum_i q_i \, \delta_D(\vec{x} - \vec{x}_i(t))$$

$$\Rightarrow - \int \sum_i A_j \, q_i \, \delta_D(\vec{x} - \vec{x}_i(t)) \, dx^j \, d^3\vec{x}$$

$$\Rightarrow \sum_i \int A_j \, q_i \, dx^j \Rightarrow \sum_i q_i \int A_j \, dx^j$$

(22) Argument for why
$\rho(t, \vec{x}) = \sum_i q_i \, \delta_D(\vec{x} - \vec{x}_i(t))$ is the charge Density?



Density is defined at particular t
$\therefore$ δt is a spatial

(23) $-\int A_j \rho \, dx^j \, d^3\vec{x} = -\int A_j \rho \frac{dx^j}{dt} d^4\vec{x}$

This is a 4-vector.

Scalar
$\nwarrow$
d^4x
$\uparrow$

$\boxed{\underline{\text{Def}}: \ \underline{\text{Current 4-vector}} \quad J^i = \rho \frac{dx^i}{dt}}$ $\rightarrow$ $dq = \rho \, dV$

$dq \, dx^i = \rho \, dV \, dx^i = \rho \, dV \frac{dx^i}{dt} dt$

Inv $\swarrow$ $\downarrow$ 4vector

$\therefore \rho \frac{dx^i}{dt}$
$\swarrow$
4vector

$\therefore -\int A_j \rho \frac{dx^j}{dt} d^4\vec{x} = -\int A_j J^j d^4\vec{x}$

(24) ~~J^i is the 4 vector BCZ as ρ is scalar dx^i transf- like 4-vector~~

$dt = \frac{ds}{\gamma}$

Imp.

(12) (25) $J^i \equiv \rho \frac{dx^i}{dt} \equiv (\rho, \rho\vec{v})$

(26) <u>Important</u>

Do not think $J^i \neq \gamma\rho u^i$

Bcz ρ is 3 Dim. Density by (22)

$J^i = \rho \frac{dx^i}{dt}$ combination of ρ, dt makes J^i 4 vector.

(27) Why J^i is 4-vector ?

(28) $J^i = (\rho, \rho\vec{v}) \equiv (\rho, \vec{J})$

(29) $\therefore A = -m\int ds - \int A_j J^j d^4\vec{x}$

<u>for many charges</u>

(30) Action for interaction B/W EM field & Current

$$\boxed{A_{int} = -\int A_k J^k d^4\vec{x}}$$

Do Gauge Transfn

$$A_{int} \to -\int \partial_k f \, J^k d^4\vec{x}$$

$$= A_{int} - \underbrace{\int \partial_k (f J^k) d^4\vec{x}} + \int f \partial_k J^k d^4\vec{x}$$

(31) $-\int \partial_K (f J^K)\, d^4\vec{x}$

<u>Gauss Theorem</u>

It will give me a Surface ~~term~~ contribution this qty.

(32) Now as Gauge Transf^n is in my hand

$\bar{A}_j = A_j + \partial_j f.$

I can choose f s.t. it vanishes on the Surface But inside volume it is arbitrary.

$\therefore\ -\int \partial_K (f J^K)\, d^4\vec{x} = 0$

(33) $\Rightarrow$ ~~$A_{int} + \int f(\partial_K J^K)\, d^4x$~~

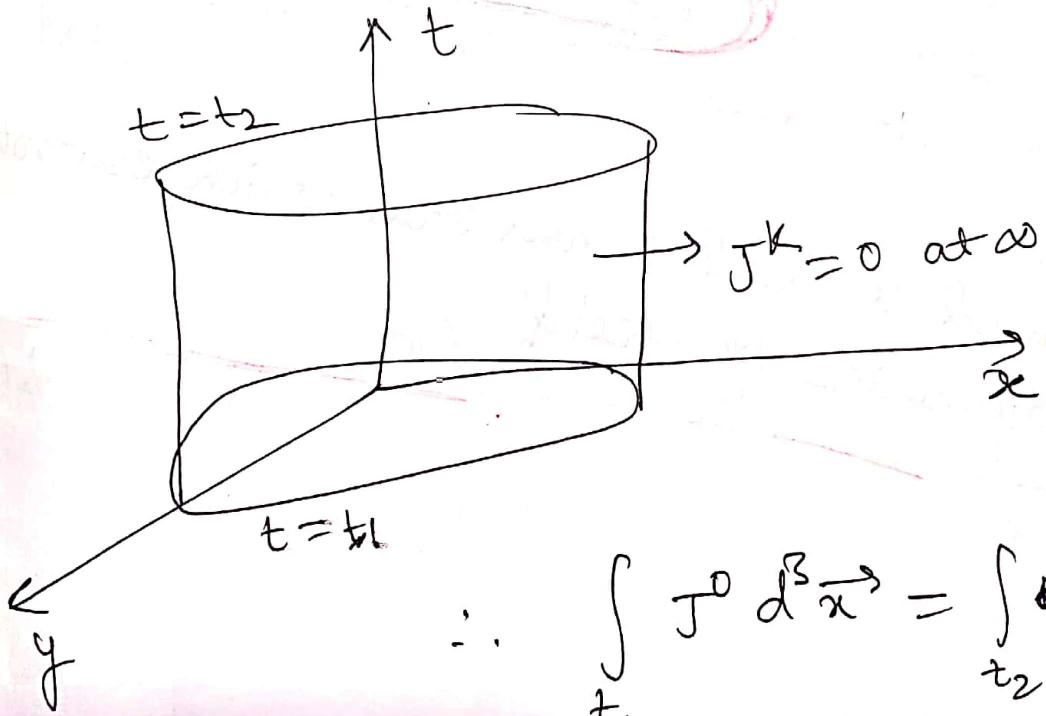
~~Demand : Action should be Gauge invariant~~

$f \neq 0$ But $\partial_K J^K = 0$

$\therefore\ \partial_K J^K = 0$

from L-(1) (32)

assuming J^K vanishes at ∞



$J^K = 0$ at ∞

$$\therefore \int_{t_1} J^0 d^3\vec{x} = \int_{t_2} J^0 d^3 x$$

(12) (34) $\therefore \int_t J^0 \, d^3\vec{x} \Rightarrow \int_t \rho \, d^3x$ Remains Same

$\therefore$ Charge Conservation.

$\boxed{\therefore \text{ Gauge Transf}^n \Longrightarrow \text{ Charge Conservation}}$

(35) Another way of getting charge conservation.

This shows that vector field can be coupled to only conserved current if G.I. is to be respected

$\partial_k J^k = 0$

$\left.\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0\right\}$ $\therefore$ charge conservation

?? $\frac{d\rho}{dt} =$

(36) All over QFT, Gauge Transf^n yields conserved charge.

(37) Action for EM field

If we think charges moving in ext. EM field close system, we have

But if we want a to think of EM field also as dynamical Entity.

$\therefore$ we need to add action term which control the dynamics of field.

~~(38)~~

58

NR Point Mech

1) Dynamical Variable

$q(t)$

Independent Variable t

Dependent Variable q

2) Lagrangian (closed system)

$L(q, \dot{q})$

$\dot{q} \equiv \partial_0 q$

In Relativity one cannot treat time coordinate preferentially in a Lorentz invariant manner.

3) Action $= \int_{t_1}^{t_2} L\,dt$

↑ Ind. Variable

4) $\delta A = \int dt \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right\} \delta q$

$+$

$\int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right)$

SR field theory

1) Dynamical Variable

Scalar field

if $\phi'(t') = \phi(t, \vec{x})$

in Newton $\phi'(t') = \phi(t)$

$\therefore \phi(t, \vec{x})$

Independent variable t, x, y, z

Dep. Var ϕ

2) Lagrangian (closed system)

$L(\partial_a \phi, \phi)$

all derivatives of ϕ

Bcz if $\frac{\partial \phi}{\partial t}$ only then in some other coordinate system $L(\partial_{a'} \phi', \phi')$

$\therefore L(\partial_a \phi, \phi)$

3) Action

$A = \int L(\partial_a\phi, \phi)\, d^4\vec{x}$

$t, \vec{x} =$ Ind. variable

No ds can be defined for field

as ds is defined for particle traj

4) $\delta A = \int d^4x \left\{ \frac{\partial L}{\partial \phi} - \partial_a \pi^a \right\} \delta\phi$

$+$

$\int d^4x\, \partial_a(\pi^a \delta\phi)$

(39) Variation for Action of EM field

Why Action for particle not considered

$$\delta A = \int d^4x \left\{ \frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_a \phi)} \delta(\partial_a \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_a \phi)} \partial_a(\delta\phi) \right\}$$

$$\frac{\partial L}{\partial(\partial_a \phi)} \partial_a(\delta\phi) = \partial_a\left(\frac{\partial L}{\partial(\partial_a\phi)} \delta\phi \right) - \left\{ \partial_a \left(\frac{\partial L}{\partial(\partial_a \phi)} \right) \right\} \delta\phi$$

$$\delta A = \int d^4x \left[\left\{ \frac{\partial L}{\partial \phi} - \partial_a \left(\frac{\partial L}{\partial(\partial_a \phi)} \right) \right\} \delta\phi \right] + \int d^4x \, \partial_a \left(\frac{\partial L}{\partial(\partial_a \phi)} \delta\phi \right)$$

Def : $\frac{\partial L}{\partial(\partial_a \phi)} \equiv \pi^a$

π^a generalizes $\frac{\partial L}{\partial(\partial_0 \phi)} = \frac{\partial L}{\partial \dot{q}}$

π^a is Analogous to Canonical mom.

(40) $$\therefore \delta A = \int d^4x \left\{ \frac{\partial L}{\partial \phi} - \partial_a \pi^a \right\} \delta\phi + \int_V d^4x \, \partial_a(\pi^a \delta\phi)$$

(41) Boundary Term

$$\int_V d^4x \, \partial_a(\pi^a \delta\phi) = \int_{\partial V} d^3x \, (n_a \pi^a) \delta\phi$$

Gauss Theorem

Assuming at ∞ : $\pi^a = 0$

$\therefore$ at $t = t_1, t_2$

$$\int_{\partial V} d^3x \, (n_0 \pi^0) \delta\phi = \int_{\partial V} d^3x \, \pi^0 \delta\phi = \int_{\left(\substack{t=t_2 \\ \text{Surf}}\right)} d^3x \, \pi^0 \delta\phi - \int_{\left(\substack{t=t_1 \\ \text{Surf}}\right)} d^3x \, \pi^0 \delta\phi$$

$\therefore$ To keep B.T $\Rightarrow 0$ let $\delta\phi = 0$

(42) $H = p\dot{q} - L = \left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - L$ (for particle)

what would be $\mathcal{H}$ for field?

(43) Def: Energy Momentum Tensor T^a_b

$$T^a_b = \pi^a \partial_b \phi - \delta^a_b L \Rightarrow \text{Compare with } H$$

in eqn then δ^a_b is Unit matrix

Think of it like if we are having matrices

It is known E-M Tensor Bcz T^0_0 would give Energy Density $T^0_0 = \pi^0 \dot{\phi} - L = p\dot{q} - L$

&

T^0_α would given momentum, $\alpha = 1,2,3$. just as continuity Eqn

$p \longrightarrow \pi$
$\dot{q} \longrightarrow \partial\phi$
In general :. 2 indices
$\pi^a \partial_b \phi$

Th:

(44) $\boxed{\partial_a T^a_b = 0 \Longleftrightarrow \text{Field Equations.}}$

(45) Field theoretic Eqn for EM (Now there is a vector field A_j as Dynamic variable)

$$A = -m\int ds - \int A_k J^k d^4x + \text{field Action}$$

In principle, we can get something more general However due to superposition

Now what would be L?

$A_f \propto \int d^4x \; L$

in NR $L \propto v^2$ quad. in first deriv. of dynamic variable

(46) Experimentally, EM theory obeys superposition Principle.

$\therefore$ field Eqn has to be linear in field variables

&

field Eqn has to be 2nd order in time.

Why?

$L(q, \dot{q}) \rightarrow$ gives 2nd order EOM

as Action has to be L.I $\Rightarrow L$ is quadratic in first derivative of dyn. var

(47) $A \propto \int d^4x \; L(\partial_j A_k)$ in a quad fashion of dyn. variable

(48) $\boxed{\text{We want Action to be Gauge invariant.}}$
& as we have seen A_j can't be determined but $F^a{}_b$ can be.

$\therefore$ we want dynamic Eqn to determine F not A_j

$\therefore \; A \propto \int d^4x \; L(F_{ab})$

Quad-ratic fn of F_{ab}
&
Action to be Gauge invariant

Assumptions

(49) (1) Superposition Principle $\Rightarrow$ Field Eqn linear in field variables

(2) Gauge invariant Action $\Rightarrow$ $\therefore$ Only Gauge invariant elements should be present
& the only G.I. having first derivative of 4-vector Potential is F_{ab}.

(3) Action has to be Lorentz invariant.

(50) What are the Quadractic functionals of F_{ab} as Lag. has to be ~~scalar~~ invariant?

L-(5) 8, 9, 11

By Th only 2 are possible

(1) $F^{ab} F_{ab}$

(2) $\epsilon^{abcd} F_{ab} F_{cd}$

(51) $$\epsilon^{abcd} F_{ab} F_{cd} = 2\epsilon^{abcd} F_{ab} \partial_c A_d$$

$$= \partial_c \left(2\epsilon^{abcd} F_{ab} A_d\right) - 2\epsilon^{abcd} \partial_c (F_{ab}) A_d$$

$$= \partial_c \left(2\epsilon^{abcd} F_{ab} A_d\right) - 2\epsilon^{abcd} \partial_c (\partial_a A_b - \partial_b A_a) A_d$$

$$= \partial_c \left(2\epsilon^{abcd} F_{ab} A_d\right) - 2\epsilon^{abcd} (\partial_c \partial_a A_b) A_d + 2\epsilon^{abcd} (\partial_c \partial_b A_a) A_d$$

$$A^{ac} S_{ac} = 0$$

$$\therefore \epsilon^{abcd} F_{ab} F_{cd} = \partial_c \left(2\epsilon^{abcd} F_{ab} A_d\right)$$

(52) If (2) $\epsilon^{abcd} F_{ab} F_{cd}$ is the Quadratic form of $L(F_{ab})$ then

$$A \propto \int_{\partial V} d^3x \, 2\epsilon^{abcd} n_c F_{ab} A_d \quad \therefore \delta A \propto \delta \int_{t=t_1}^{t=t_2} d^3x \, 2\epsilon^{abcd} F_{ab} A_d$$

Assuming fields vanish on ~~field~~ space like surface & on timlike surface we are keeping it frozen $\therefore$ No variation ~~δA on Boundary = 0~~

$\therefore \delta A = 0$

$\therefore$ (2) is not useful to put in $L(F_{ab})$

↳ in Topological QFT only Surface terms will contribute

We don't want action which is pure surface term. We want Action which is $\int d^4x$ (which can vary in space & time)

(53) ∴ Suppose we have a Surface term & we are going to vary it assuming at Surface it is not changing

⟹ then they contribute to my field eqn:-

(54) ∴ $A_c \propto \int d^4x \, F^{ab} F_{ab}$.

$$A_c = \frac{-1}{16\pi} \int d^4x \, F^{ab} F_{ab}$$

$$\frac{1}{4\pi} \xrightarrow{SI} \epsilon$$

(55) why -ve sign?

$$F^{ab} F_{ab} = 2 (B^2 - E^2)$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

↑ energy term. $\left(\frac{\partial A}{\partial t}\right)^2$?

& we want this $\frac{\partial A}{\partial t}$ to have +ve sign.

∴ (-ve sign) ahead. } why?

(56) ∴ $A = -m \int ds - \int A_k J^k d^4x - \frac{1}{16\pi} \int d^4x \, F^{ab} F_{ab}$

Vary Action w.r.t. vector Potential. ?

We are not touching charges

∴ $-m \int ds$ doesn't change & J^k remains same

57 Now A_k is like a scalar field as field is the dynamic variable as we are varying it.

58 $$\delta A = -\int d^4x\, J^k \delta A_k - \frac{2}{16\pi}\int d^4x\, F^{ab}\delta(F_{ab})$$

$$F^{ab}\delta(F_{ab}) = 2F^{ab}\delta(\partial_a A_b)$$
$$= 2F^{ij}\partial_i(\delta A_j)$$
$$= 2\partial_i(F^{ij}\delta A_j) - 2\partial_i(F^{ij})\delta A_j$$

$$\therefore\ \delta A = -\int d^4x\, J^k \delta A_k - \frac{1}{4\pi}\int d^4x\left\{\partial_i(F^{ij}\delta A_j) - \partial_i(F^{ij})\delta A_j\right\}$$
$$= -\int d^4x\, J^k\delta A_k - \frac{1}{4\pi}\int d^4x\left\{\partial_i(F^{ik}\delta A_k) - \partial_i(F^{ik})\delta A_k\right\}$$

$$= -\int_V d^4x\, \delta A_k\left\{J^k - \frac{1}{4\pi}\partial_i F^{ik}\right\} - \frac{1}{4\pi}\int_V d^4x\, \partial_i(F^{ik}\delta A_k)$$

Compare 39

Gauss Theorem

$$\int_{\partial V} d^3x\, n_i F^{ik}\delta A_k$$

Assuming δA_k on surface vanishes

59 $\therefore$ <u>Field Equations</u>

$$\partial_i(F^{ik}) = 4\pi J^k \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \text{ Cont. Eqn}$$

$$\partial_k\partial_i F^{ik} = 4\pi\partial_k J^k$$
Asy sy
$$0 = \partial_k J^k$$

If $k=0$ $\quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho \xrightarrow{\text{in SI}} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

(53) $\vec{\nabla} \times \vec{B} = 4\pi \vec{j} + \frac{\partial \vec{E}}{\partial t} \Rightarrow$ conservation of electric charge. & as eq. I. is assumed in making $L = -\frac{1}{16\pi} F_{ab} F^{ab}$

(60) ∴ Maxwells Eqn

$$= \quad \partial_i F^{ik} = 4\pi J^k \quad \text{(source Path)}$$

$$\partial_i F^{*ik} = 0 \quad \text{(source free Path)}$$

(54)

(61) $$\delta A_c = -\frac{1}{4\pi} \int_V d^4x \, \partial_i (F^{ik} \delta A_k)$$

$$= -\frac{1}{4\pi} \int_{\partial V} d^3x \, n_i F^{ik} \delta A_k$$

$$\Rightarrow \quad = -\frac{1}{4\pi} \int_{\substack{\text{const} \\ t=\text{surf}}} d^3x \, n_0 F^{0k} \delta A_k$$

$$= -\frac{1}{4\pi} \int_{\substack{\text{const} \\ t=\text{surf}}} d^3x \, F^{0k} \delta A_k$$

(29) L-(4)

in F matrix spatial parts are of $\vec{E}$.

~~Only~~ Spatial part Contributes

as $F^{00} = 0$

$$\therefore \delta A_c \propto \int d^3x \, (\vec{E} \cdot \delta\vec{A})$$

(62) $\boxed{\begin{array}{l} \text{fix } \delta\vec{A} \text{ on surface} \\ \therefore \text{ Not fix } \delta\phi \text{ (scalar pot)} \\ A^i = (\phi, \vec{A}) \end{array}}$

EM

$A'_\mu \rightarrow A_\mu + \frac{\partial f}{\partial x^\mu}$

f is in my hand i.e. freedom

choose f s.t. A'_0 is zero

i.e. choose f s.t. in one frame A_0 is zero

$\therefore \quad 0 = A_0 + \frac{\partial f}{\partial t} \Rightarrow \boxed{\frac{\partial f}{\partial t} = -A_0}$ — (1)

Can I choose such f.

By (1) Yes
Eqn

coz (1) soln $\exists$

FIXING THE GAUGE

$\therefore$ We have 3 Ind. comp. of A instead of 4

$\left.\begin{matrix} A_0 = 0 \\ A_m(x^i) \end{matrix}\right\}$ Degree of freedom space comp. of A

$E = -\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi = -\frac{\partial \vec{A}}{\partial t}$

$B = \nabla \times A$

~~Degrees of~~

$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \vec{A}}{\partial t}\right)^2 - \frac{(\nabla \times A)^2}{2} \equiv \left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{(\partial_x \phi)^2}{2}$

$\pi_x = \frac{\partial \mathcal{L}}{\partial(\partial_t A_x)} = \frac{\partial A_x}{\partial t}$ $\quad \underbrace{A_x \quad A_y \quad A_z}$

$\pi_y = \partial A_y / \partial t \quad \pi_z = \frac{\partial A_z}{\partial t}$

Each has its own canon. Mom

(1, (53) (4) But $\vec{E} = -\frac{\partial A}{\partial t}$

$\therefore \pi_x = -E_x$ $\Big\}$ Bec. $\delta A_c = \int_{\partial V} d^3x\, \vec{E}\cdot\delta\vec{A} = -\vec{p}$

(5) $H = \frac{1}{2}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{(\nabla\times A)^2}{2} = k.E + P.E$

(54) $= \frac{E^2+B^2}{2} \geq 0$

(6) For EM wave

$|E| = |B|$

$\therefore H = E^2$

(55) (7) Momentum Density

$P = \int d^3x\, \pi \frac{\partial\phi}{\partial x}$ (scalar field)

$= \int d^3x\, \pi\, \delta\phi$

for Multiple fields ϕ_i

$P = \sum_i \int d^3x\, \pi_i\, \delta\phi_i$

for EM

(56) $P_n = \sum_m \int d^3x\, E_m \frac{\partial A_m}{\partial x^n}$

n^{th} component

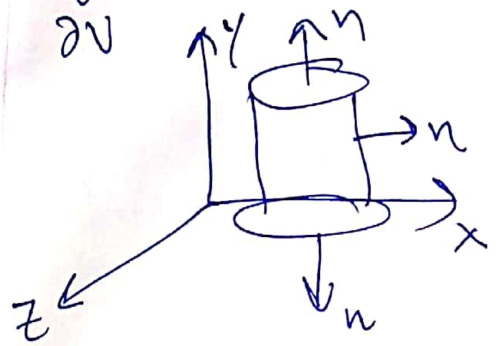
(8) $P_n = \sum_m \int d^3x\, E_m \left[\frac{\partial A_m}{\partial x^n} - \partial_m A_n\right] + \underbrace{\sum_m \int d^3x\, \partial_m A_n E_m}$

$= \int d^3x\, E_m F^m_n + \sum_m \int d^3x\, (\partial_m A_n) E^m$

$$\int d^3x\, (\partial_m A_n) E^m$$

$$= \int d^3x\, \partial_m (A_n E^m) - \int d^3x\, A_n \partial_m E^m$$

$$\int_{\partial V} d^2x\, A_n E^m n_m - \int d^3x\, A_n \partial_m E^m$$



Assuming E_m vanish at infinity and E_m vanish at Bdry.

$$= -\int d^3x\, (\partial_m E^m) A_n$$

Assuming free field.

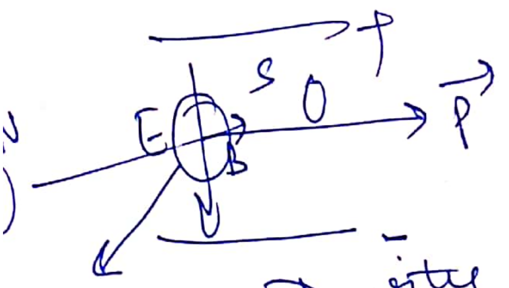
$$= 0$$

$$P_n = \sum_m \int d^3x\, E_m \left[F^m{}_n \right]$$

$$P_1 = \int d^3x\, (E_y B_z - E_z B_y)$$

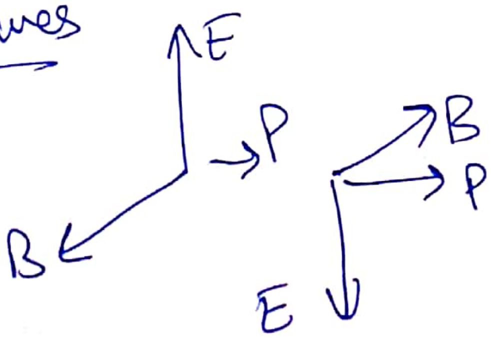
$$\therefore \vec{P} = \int d^3x\, \underbrace{(\vec{E} \times \vec{B})}_{}$$

Poyting vector



Momentum Density

EM Waves



→ Poynting vector

(1 53 10

Energy Density   Density P_x Density P_y Den. P_z

$$T^{\nu\mu} = \begin{matrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & & \\ T^{30} & T^{31} & & \end{matrix}$$

flux of Energy in x

flux of Energy in y

54

x, y, z Flux of x mom.

f_0 f^m

Density   flux

55

56

2) Our dynamical variables were $A_j = (\phi, \vec{A})$ (69)

But if our Dynam variable were $q_1, q_2 \dots q_{10}$

$\therefore$ δA calculating it, we would have to fix all of them at end pts so as to vanish that term

But in this case ϕ doesn't have to be fixed.

Dynamical variable is something whose time derivative comes in quadratic manner in action.

Why? Time derivative can only come if one of the indices is 0. $F_{ab}F^{ab}$ — We can think of it as $E^2 - B^2$; B doesn't have time derivatives; ~~E has time derivative of vector~~ pote.

If one of them is 0 then other has to be space. Time derivative of scalar potential ϕ doesn't come in the action.

$\therefore$ Only time derivative of $\vec{A}$ comes.

$\therefore$ ϕ is vague kind of dof.

5) As Action is Gauge invariant it is possible to choose $\phi = 0$.

$\boxed{\bar{\phi} = \phi + \frac{\partial f}{\partial t}}$ not any time derivative as $B^2 - E^2$; E has $\frac{\partial \vec{A}}{\partial t}$

6) We don't have to fix whole $\vec{A}$. I can't even see $\vec{A}$, how can I fix it. I can fix $\vec{A}$ only up to Gauge transfn.

$\int d^3x\, \vec{E} \cdot (\delta \vec{A}) = \int d^3x\, \vec{\nabla}(\vec{E} \cdot \vec{f}) - \int d^3x\, \vec{\nabla} \cdot \vec{E}\, \vec{f}$

$\delta \vec{A} = \vec{\nabla} f$

vanish. Why

$\vec{\nabla} \cdot \vec{E}$: free field 0, otherwise $\frac{\rho}{\epsilon_0}$ which we can ignore

$\equiv \int d\left(\frac{\partial L}{\partial \dot{q}} \delta q\right)$

$\therefore$ We have to fix A only w.r.t. Gauge Transf.

As we only need to fix the part of $\vec{A}$ which is unaffected by Add $\vec{\nabla} f$ i.e. $\vec{B} = \vec{\nabla} \times \vec{A}$ $\therefore$ we have to fix only B

$\vec{B}$ are coordinates in EM

$\vec{E}$ are momenta in EM.

Bec in CM only coord x is fixed at end pt. Here B is fixed.

(68) $\underbrace{\partial_k (\partial_i F^{ik})}_{0} = 4\pi \partial_k J^k$

$\therefore$ J^k is conserved — See L-5 (33) (34) (35)

(69) when
$A = -m\,ds - \int A_k J^k d^4\vec{x}$

A_k is specified

when

$A = -m\,ds - \int A_k J^k d^4\vec{x} - \frac{1}{16\pi}\int d^4\vec{x}\, F_{ab}F^{ab}$

J is externally specified.

If J^k is not conserved then we get to eq. $\partial_i F^{ik} = 4\pi J^k$ & get into contradiction (68)

(70) $\therefore$ Gauge invariant field can only couple to a source which is conserved.

Scalar field | EM field