L-11

(64)

① To check if the spacetime is flat or not

① coord. Transf.

② check if all $R = 0$ then flat.

② in (57)

RHS need not be zero

if it is not zero then $\partial_i k^j = -\Gamma^j_{li} k^l$ do not posses a soln

∴ we can't // transp. globally

But if I have given curve, then I can // transp. along that curve.

Bcz. given a vector & curve, we can // transp. along that curve always.

③ **Th.** If spacetime is flat $\Rightarrow R = 0$

Proof. if the spacetime is flat we can always choose a inertial coordinate system in which the Γ vanish at all events.

∴ The derivatives of Γ will also vanish making $R = 0$

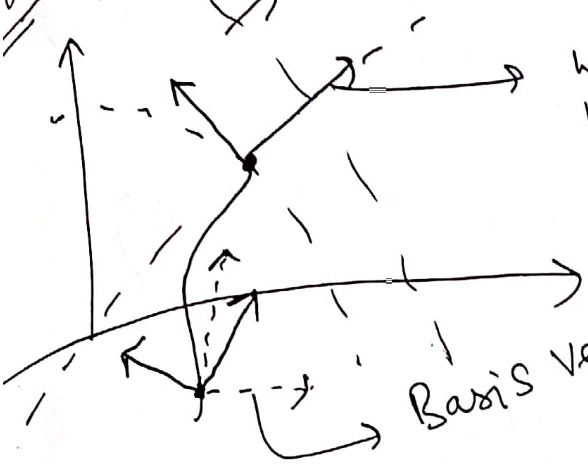
Imp. But as R is Tensor & ∴ if it vanish in one coordinate system then it has to vanish in any other coord. system.

∴ in flat spacetime $R = 0$.

④ If $R = 0$ ~~then~~ $\partial_i K^j + \Gamma^j_{ki} K^l = 0$ has ~~soln~~ unique soln.

~~then space-time is flat.~~

ii → Polar coord.



We can transport everywhere this basis & we get cartesian grid.

Cartesian grid would be rotated w.r.t. original Basis at every pt.

Basis vector given.

Th ⑤ If $R \neq 0$ then $\partial_i K^j + \Gamma^j_{li} K^l = 0$ doesn't have soln.

⇒ ⑥ & ⑦ are equivalent

If I can ∥ transport globally then I can get Cartesian coordinates globally ∴ Transf. to global Cartesian coordinates can be done.

Th ⑥ If I can ∥ transf. globally ⇒ spacetime is flat.

Corollary: If $R=0$ ⇒ spacetime is flat.

Th ⑦ Transfⁿ to global cartesian coord ⇒ Spacetime is flat

Th ⑧ If spacetime is flat ⇒ transf. to global Cart. coord

⑨ $\nabla_i \nabla_j$ covariant derivative does not commute like $\partial_i \partial_j$. i.e. $\nabla_i \nabla_j v^k - \nabla_j \nabla_i v^k = (\nabla_i \nabla_j - \nabla_j \nabla_i) v^k \neq 0$

⑩ $\nabla_i(\nabla_j v^k) = \partial_i(\nabla_j v^k) + \Gamma^k_{\cancel{l}i}\nabla_j v^{\cancel{l}} - \Gamma^l_{ij}\nabla_l v^k$

$= \partial_i\partial_j v^k + \partial_i(\Gamma^k_{mj}v^m) + \Gamma^k_{li}(\partial_j v^l + \Gamma^l_{mj}v^m)$

$- \Gamma^l_{ij}(\partial_l v^k + \Gamma^k_{ml}v^m)$

$(\nabla_i\nabla_j - \nabla_j\nabla_i)v^k = \partial_i(\Gamma^k_{mj}v^m) - \partial_j(\Gamma^k_{mi}v^m)$

$+ \Gamma^k_{li}(\partial_j v^l + \Gamma^l_{mj}v^m)$

$- \Gamma^k_{lj}(\partial_i v^l + \Gamma^l_{mi}v^m)$

$= (\partial_i\Gamma^k_{mj})v^m + \cancel{(\Gamma^k_{mj})\partial_i v^m} - (\partial_j\Gamma^k_{mi})v^m - \cancel{(\Gamma^k_{mi})\partial_j v^m}$

$+ \Gamma^k_{li}\cancel{\partial_j v^l} + \underline{\underline{\Gamma^k_{li}\cancel{\Gamma^l_{mj}}v^m}} - \cancel{\Gamma^k_{lj}\partial_i v^l} - \cancel{\Gamma^l_{mi}v^m\Gamma^k_{lj}}$

All terms sym in i & j would cancel out

$= (\partial_i\Gamma^k_{mj} - \partial_j\Gamma^k_{mi})v^m + \cancel{\Gamma^k_{mi}\partial_j - \Gamma^k_{mi}}$

$(\Gamma^k_{li}\Gamma^l_{mj} - \Gamma^l_{mi}\Gamma^k_{lj})v^m$

$= (\partial_i\Gamma^k_{mj} - \partial_j\Gamma^k_{mi} + \Gamma^k_{li}\Gamma^l_{mj} - \Gamma^l_{mi}\Gamma^k_{lj})v^m$

$= R^k{}_{mij}\,v^m$

$(\nabla_i\nabla_j - \nabla_j\nabla_i)v^k = R^k_{mij}v^m$

$(\partial_i\partial_j - \partial_j\partial_i)v^k = 0$

↳ see (57)

⑪ **Th.** $R^k{}_{mij}$ is the Tensor

Proof $\nabla_i\nabla_j v^m$ is the ~~covariant~~ Tensorial obj.

∴ $(\nabla_i\nabla_j - \nabla_j\nabla_i)v^m$ is ~~covariant~~ Tensor

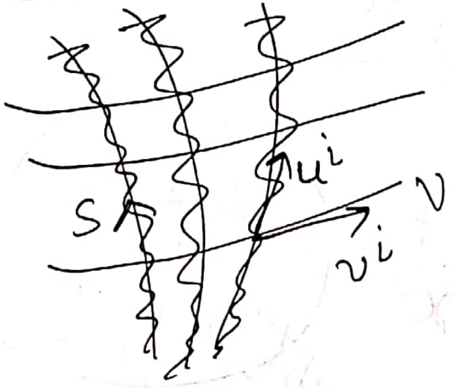
As v^m is a vector

∴ $R^k{}_{mij}$ has to be Tensor

(12) As R^{k}_{mij} is a Tensor
if it vanishes in one frame, it will vanish in all other frames. see (3)

(13) Just like we have 2nd derivative of curve being Curvature
Here also 2nd derivative of metric is curvature.

(14) Geodesic Deviation



} Geodesic

s : tells where on the geodesic I am

v : picks up on which geodesic I am

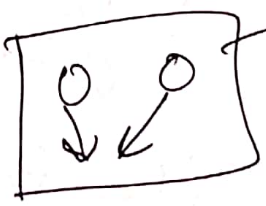
∴ s & v will tell the given location of plane.

∴ Any point on geodesic can be given as $x^i(s,v)$

(15) $u^i = \frac{\partial x^i}{\partial s}$ as now x^i depends on s & v ∴ $\frac{\partial x^i}{\partial s}$

(16) $v^i = \frac{\partial x^i}{\partial v}$: Tells the deviation B/w Geodesic

(17) If I move in u^i how will v^i change?



→ as Both follow geodesics

But by princ. of Eq. Both should have followed same path irr. of their prop?

Only valid in small reg.

⑱ In Newtonian

Rec. in plane paper

$\frac{\partial^2 v^\alpha}{\partial t^2} = 0$

$\frac{\partial v^\alpha}{\partial t}$ f(~~[illegible]~~). But It is plane of paper is what is Ref.

∴ $\frac{\partial^2 v^\alpha}{\partial t^2}$ ← We are interested in this qty. Not $\frac{\partial v^a}{\partial t}$.

$$v^\alpha = \frac{\partial x^\alpha}{\partial v}$$

$\frac{\partial^2 v^\alpha}{\partial t^2}$ = Acc. of sep. vector

$\frac{\partial v^\alpha}{\partial t}$ = vel. of sep vector

= But vel. changes in each frame in Gal. Trans. ∴ Acc. is the qty we are interested in.

⑲ $$\frac{\partial^2 v^\alpha}{\partial t^2} = \frac{\partial}{\partial v}\left(\frac{\partial^2 x^\alpha}{\partial t^2}\right) = -\frac{\partial}{\partial v}\frac{\partial \phi}{\partial x^\alpha} = -v^\beta\left(\frac{\partial^2 \phi}{\partial x^\beta \partial x^\alpha}\right)$$

$\left(1, 1, 1\right)$ ∴ $dx^\alpha = dx_\alpha$

Doubt

2nd derivative of Potential.

⑳ Tides Produced on Earth by Sun & Moon are of Equal Mag. Even Though Sun's Grav. force on Earth >> Moon's on Earth

㉑

But why water is attracted in one direction & Repelled in other?

This is due to the diff. of the force exerted at centre of Earth & water.

∴ $F \propto \frac{1}{r^2}$ $T_f = \int dF \propto \frac{d}{r^3}$ ∴ Tidal forces go by $\frac{1}{r^3}$

But Now $\frac{M_S}{r^3} \approx M_m$ ∴ Tidal forces

㉒ $$-v^\beta \frac{\partial^2 \phi}{\partial x^\beta \partial x^\beta}$$

As Tidal forces are (Grad. of forces) × Distance equivalent to

$$\frac{\partial^2 v^\alpha}{\partial t^2} = \underbrace{-v^\beta \boxed{\frac{\partial^2 \phi}{\partial x^\beta \partial x^\alpha}}}$$

→ This tells how grav. force is changing from place to place.

↳ (Grad. in β direct) × Direction in β dir.

㉓ ∴

Two Infinitesimal seperated Geodesic
then
Take force on one Geodesic
Take force on other Geodesic
Take Diff & Mult. by distance
we get acc. of geodesics towards one other.

㉔ $$\frac{\partial^2 \phi}{\partial x^\beta \partial x^\alpha}$$ ≡ This tells how grav. force is changing from place to place.

Grav force ≡ Grad. of Pot ≡ Γ

∴ we should see $\partial_i \Gamma$

㉕ We ~~could~~ want to know acc. of sep. of Geodesic.

~~$u^i \nabla_i$ = Acc.~~

$u^i \nabla_i v^j$ = change in v^j along u ~~along direction~~

$(u^i \nabla_i)(u^j \nabla_j v^k) = D^2 v^k \equiv$ Acc. of v^k along u^i

(26) Th $u^i \nabla_i v^k = v^i \nabla_i u^k$

Proof ?

$$u^i \nabla_i = \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x^k}{\partial v}\right) = \frac{\partial}{\partial v} \frac{\partial x^k}{\partial \tau}$$

(27) $u^i \nabla_i (u^j \nabla_j v^k) = u^i \nabla_i (v^j \nabla_j u^k)$

$$= u^i v^j \nabla_i \nabla_j u^k + u^i \nabla_j u^k \nabla_i v^j$$

$$= u^i v^j \nabla_i \nabla_j u^k + (\nabla_j u^k) v^i \nabla_i u^j \quad ①$$

$$(\nabla_j u^k) v^i \nabla_i u^j = v^i (\nabla_i u^j) \nabla_j u^k$$

$$= v^i \left(\nabla_i (u^j \nabla_j u^k) - (\nabla_i(\nabla_j u^k)) u^j \right)$$

But $u^j \nabla_j u^k = 0$ Geod. Eqn.

$$= -v^i u^j \nabla_i (\nabla_j u^k).$$

$$= -v^i u^j \nabla_i \nabla_j u^k$$

Putting in ①

$$= u^i v^j \nabla_i \nabla_j u^k - v^i u^j \nabla_i \nabla_j u^k$$

$$\Rightarrow u^i v^j \nabla_i \nabla_j u^k - v^j v^i \nabla_j \nabla_i u^k$$

$= v^j u^i (\nabla_i \nabla_j - \nabla_j \nabla_i) u^k$ By ⑰

$D^2 v^k = v^j u^l R^k{}_{lij} u^l$

↳ Geodesic Deviation Acc.

㉘ In Newtonian Unit — see L·⑭ ⑤

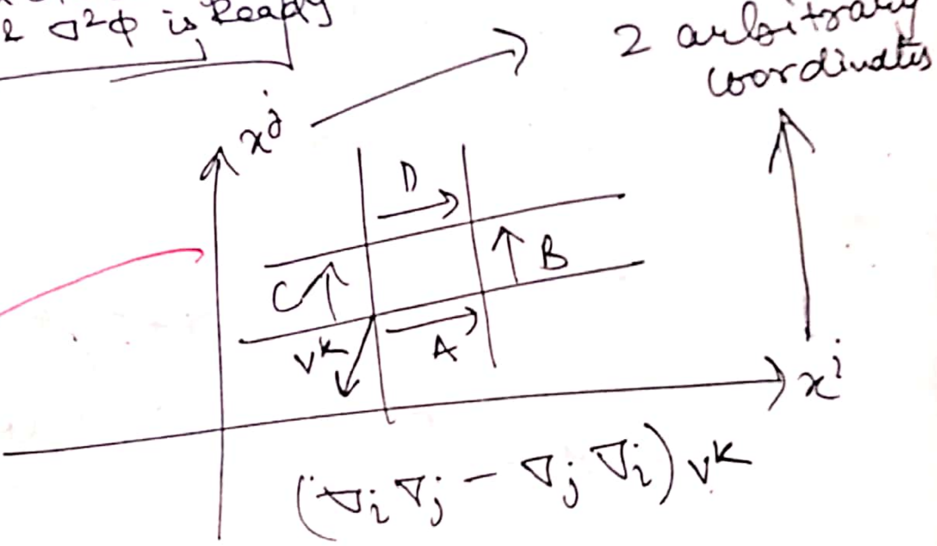
u^l, u^i Only zeroth comp. will contribute

$R_{\alpha 0 \beta 0} \simeq \partial_\alpha \partial_\beta \phi$ for ~~$R_{\alpha 0 \beta 0}$~~ $\cong g_{00} = (1+2\phi)$

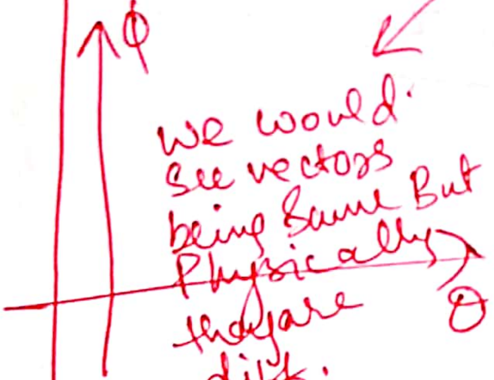
$g_{\alpha\beta} = \delta_{\alpha\beta}$

as in Newton $\nabla^2 \phi = 4\pi G \rho$
∴ from struct. of R we can tell that
we can contract it with other object
$R_{\alpha 0 \beta 0} g^{\alpha\beta} = g^{\alpha\beta} \partial_{\alpha\beta} \phi$ & $\nabla^2 \phi$ is ready
in fields eqn.

㉙ See Ch-5 ⑦ ⑩
Lec ⑪



$(\nabla_i \nabla_j - \nabla_j \nabla_i) v^K$



We would see vectors being same But physically they are diff.

A → B

$\nabla_j \nabla_i v^K$

C → D

$\nabla_i \nabla_j v^K$

$(\nabla_j \nabla_i - \nabla_i \nabla_j) v^K =$ Diff in v^K when ~~[illegible]~~ 2 paths are taken

$= R^K{}_{mij} V^m$

= Depends on Curvature

$\Delta v^a = -\frac{1}{2} (R^a{}_{bcd})_p v_p^d \oint v^c dx^d$

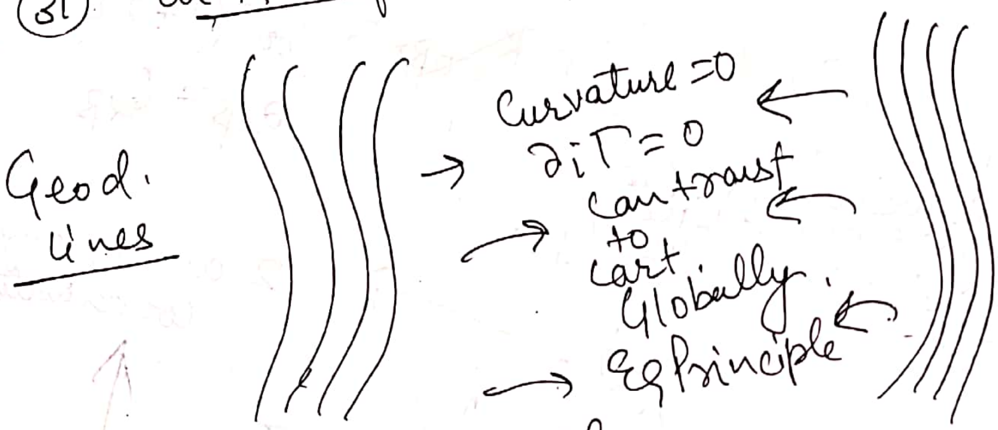
㉚ for //gm.

$$\Delta v^a = -\frac{1}{2} R^a{}_{bcd}\, v^b_p\, \underset{\uparrow}{\Delta\sigma^{cd}}$$

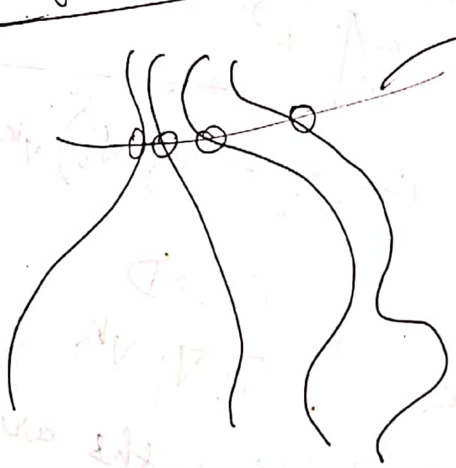
Area enclosed by the loop.

Since all others qty are tensors, $R^a{}_{bcd}$ is a tens as long as we are dealing with infinitesimal region

㉛ In Acc frame In Uniform Grav.



In Dynamic Grav. field



→ Γ at each pt different

∴ Geod. Eqn different

㉜ Properties of R

$$R^k{}_{ija} = (\partial_i\Gamma^k_{ja} - \partial_j\Gamma^k_{ia}\,\Gamma^e_{aj} - \Gamma^k_{je}\Gamma^e_{ai})$$

$$\boxed{R_{mija} = g_{mk} R^k{}_{ija}}$$

㉝ Around Local inertial frame at any pt.

Imp $\Gamma^i_{jk} = 0$

$g_{ab} = \eta_{ab}$

But $\partial_e \Gamma^i_{jk} \neq 0$ Bec. Earlier we said in LIf.

all $g_{ab}|_p = \eta_{ab}$

$\partial_e g_{ab}|_p = 0$

But $\partial_i \partial_e g_{ab}|_p \neq 0$

except 2D.

㉞ If $g_{ab} = \eta_{ab}$ globally

then $\partial_e g_{ab} = \partial_e \eta_{ab} = 0$ & $\partial_i \partial_e \overset{\eta_{ab}}{\cancel{g_{ab}}} \overset{=}{\cancel{\neq}} 0$

But here we have to compute $(\partial_e g_{ab})|_p = 0$

& Similarly $\partial_e \partial_i g_{ab}|_p \neq 0$.

㉟ ∴ Curvature ∃ in local inertial frame. ☐ BOX

∴ There is diff. B/w local I.f. &.

㊱ $R_{maija} = g_{mk}(\partial_i \Gamma^k_{ja} - \partial_j \Gamma^k_{ia})$

$= \partial_i (g_{mk} \Gamma^k_{ja}) - \partial_j \cancel{\partial} (g_{mk} \Gamma^k_{ia}) - (\partial_i g_{mk}) \Gamma^k_{ja} + (\partial_j g_{mk}) \Gamma^k_{ia}$

$= \partial_i \Gamma_{mja} - \partial_j \Gamma_{mia}$

$= \frac{1}{2} \cancel{2} [\partial_i(-\partial_m g_{ja} + \partial_j g_{am} + \partial_a g_{mj})]$

$- \frac{1}{2}[\partial_j(-\partial_m g_{ia} + \partial_i g_{am} + \partial_a g_{mi})]$

$= \frac{1}{2}(\partial_i(-\partial_m g_{ja} + \partial_a g_{mj}) - \frac{1}{2}\partial_j(-\partial_m g_{ia} + \partial_a g_{mi})$

$= \frac{1}{2}[-\partial_i\partial_m g_{ja} + \partial_i \partial_a g_{mj} + \partial_j \partial_m g_{ia} - \partial_j \partial_a g_{mi})$

(37) Originally we had

$$R_{abcd} = -R_{abdc}$$

Now

$$R_{abcd} = -R_{bacd}$$ from (36)

(38) $R_{abcd} = R_{cdab}$ from (36)

(39) Though we are proving it in L.I. frame
But ~~they~~ ~~wou~~ properties would handle in every frame.
Because R_{abcd} are Tensors.
and hence when $R_{a'b'c'd'} = \partial_{a'}x^a \partial_{b'}x^b \partial_{c'}x^c \partial_{d'}x^d R_{abcd}$

~~(40)~~

$$R_{a'b'd'c'} = \partial_{a'}x^a \partial_{b'}x^b \partial_{d'}x^c \partial_{c'}x^d R_{abcd}$$

$$= \partial_{a'}x^a \partial_{b'}x^b \partial_{d'}x^d \partial_{c'}x^c R_{abdc}$$

$$= -\partial_{a'}x^a \partial_{b'}x^b \partial_{d'}x^d \partial_{c'}x^c R_{abcd}$$

$$\boxed{R_{a'b'd'c'} = -R_{a'b'c'd'}}$$

∴ Valid in Any frame →

$ds^2 = \int \eta_{ab} dx^a dx^b$
L.I.f.
$\eta_{ab} = g_{ab}$
$ds^2 = \int g_{ab} dx^a dx^b$
Tensor
∴ valid in any frame

(40) $R_{abcd} + R_{adbc} + R_{acdb} = 0$

$$R_{a[bcd]} = 0$$

Proof

(41) $R_{abcd} + R_{adbc} + R_{acdb} = 0$

↑

This is Totally Antisym. Tensor

Proof. $-R_{abdc} + R_{bcad} + R_{dbac} = 0$

~~$-R_{abdc} + -R_{bcda} + -R_{bd}$~~

$-R_{abdc} + R_{bcad} + R_{bdca} = 0$

$-R_{badc} + R_{acbd} + R_{adcb} = 0$

$R_{abdc} + R_{acbd} + R_{adcb} = 0$

$R_{abcd} + R_{adbc} + R_{acdb}$

$= R_{bacd} + R_{bdac} + R_{bcda}$

$= -(R_{abcd} + R_{acdb} + R_{adbc})$

(42) How many independent comp does R have? from L-7

(34)

Excess Dof. $\frac{N^2(N^2-1)}{12}$ = No. of Ind. comp.

Exc. dof = 6 in 4D

As in $g_{ab} = \eta_{ab}$ which is equal to Dof used for L.T.

$g_{ab} = \eta_{ab}$	10 cond	16 par
$\partial_j g_{ab} = 0$	40 cond	40 par
$\partial_i \partial_j g_{ab} = 0$	100 cond	80 par

$\therefore 20 = \frac{N^2(N^2-1)}{2}$ such conditions were left which were not followed

㊸ $R_{\textcircled{ab}\textcircled{cd}}$

Let pair take M ind. values.

$\therefore$ as $R_{abcd} = R_{cdab}$ (Sym)

$$ {}^{M+1}C_2 = \frac{M(M+1)}{2} $$

$M = {}^NC_2$ as $R_{abcd} = -R_{abdc}$

㊹ $R_{a[bcd]} = 0$ $\therefore$ if $R_{1123} = 0$

As This is Totally A.S.
any 2 indices same $\Rightarrow R = 0$
$\therefore$ we have to take all indices different.
$\therefore$ Independent condn will be given when all indices are diff.

see ch-6 ㉜

${}^NC_4$ i.e. ${}^NC_4$ terms can be written in other terms

㊺ Ind. Comp $= {}^{M+1}C_2 - {}^NC_4$

$$ = \frac{{}^NC_2({}^NC_2+1)}{2} - \frac{N(N-1)(N-2)(N-3)}{4.3.2} $$

$$ = \frac{N(N-1)}{2}\frac{(N(N-1)+2)}{8} - \frac{N(N-1)(N-2)(}{24} $$

$$= \frac{3N^2-3N+\cancel{6} - N^2+5N\cancel{-6}}{3}\left(\frac{N(N-1)}{8}\right)$$

$$= \frac{2N^2+2N}{3}\;\frac{(N(N-1))}{8}$$

$$= \frac{N^2(N+1)(N-1)}{12}$$

$$= \frac{N^2(N^2-1)}{12}$$

(46) In $N=4$

R has 20 Ind. Comp.

(47) In Newtonian

R is 3X3 Sym. Matrix

$\therefore$ 6 Ind. Comp.

→ In 2 D

1 Ind Comp

(48) R^{B}_{ijkl}

Th Only contraction on i & k is non Trivial.

See Ch-⑥ 34, 35

$R = R_{jl}g^{jl} = R_{lj}g^{jl}$

$R_{jl} = R_{lj}$

(49) $g^{ik}R_{ijkl} = R_{jl} \equiv$ Ricci Tensor → 10 Ind comp

$R = R_{jl}g^{jl}$

Imp

-3) (50) R_{jl} is Sym. in j & l.

Proof: $R_{jl} = g^{ik}R_{ijkl} = g^{ik}g_{im}(\partial_k\Gamma^m_{jl} - \partial_l\Gamma^m_{jk} + \Gamma^m_{ok}\Gamma^o_{jl} - \Gamma^m_{ol}\Gamma^o_{jk})$

(51) $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ = Einstein Tensor

↳ Sym in a & b.

L-12

① **Bianchi Identity**

$$\nabla_i R^a{}_{bcd} + \nabla_c R^a{}_{bdi} + \nabla_d R^a{}_{bic} = 0 \quad — ①$$

Proof $R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$

$$\nabla_i R^a{}_{bcd} = \nabla_i(\partial_c \Gamma^a_{bd}) - \nabla_2(\partial_d \Gamma^a_{bc}) + \Gamma^e_{bd}\nabla_i\Gamma^a_{ec} + [\nabla_i \Gamma^e_{bd}]\Gamma^a_{ec} - \Gamma^a_{ed}\nabla_i\Gamma^e_{bc} - \Gamma^e_{bc}\nabla_i\Gamma^a_{ed}$$

But in local Inertial frame

$g_{ij} = \eta_{ij}$

$\partial_i g_{ab} = 0 \Rightarrow \Gamma^i_{jk} = 0$

$$\therefore \nabla_i R^a{}_{bcd} = \nabla_i(\partial_c \Gamma^a_{bd}) - \nabla_i(\partial_d \Gamma^a_{bc})$$
$$= \partial_i(\partial_c \Gamma^a_{bd}) - \partial_i(\partial_d \Gamma^a_{bc})$$

from this ① can be easily proved.

② Raising b in R $\quad R^{ab}{}_{cd}$

Doubt

In $R^a{}_{bcd}$ if raising b in Antisymm. there can be a sign flip.

In $R^{ab}{}_{cd}$ we have to raise b in Γ which is not easy.

③ $\nabla_i R^{ab}{}_{cd} + \nabla_c R^{ab}{}_{di} + \nabla_d R^{ab}{}_{ic} = 0$ 15.6

$\Rightarrow (\nabla_i R^{ab}{}_{cd}) g^c_a = (\nabla_i R^{ab}{}_{cd} g^c_a) - \underset{\overset{\|}{0}}{(\nabla_i g^c_a)} R^{ab}{}_{cd}$

$\nabla_i (R^{ab}{}_{cd} g^c_a) = \nabla_i R^b{}_d \longrightarrow g\delta^i_b$

$\Rightarrow g^i_b \nabla_i R^b{}_d = g_{bk} g^{ki} \nabla_i R^b{}_d$

$= g_{bk} \nabla^k R^b{}_d$

$= \boxed{\nabla_b R^b{}_d}$

$\boxed{\begin{aligned} & g^a_b = \delta^a_b \\ \text{Proof } & g^a_b = g^{am} g_{mb} \\ & = \delta^a_b . \end{aligned}}$

$\therefore \cancel{\nabla_b R^b{}_d}$

$\cancel{\nabla_c (R^{ab}{}_{di}) = \nabla_c (R^{ab}{}_{di} g^c_a) - (\nabla_c g^a_a) R^{ab}{}_{di}}$

$\cancel{= \nabla_c (R^{ab}{}_{di} g^c_a) - \frac{\partial_c(\sqrt{-g}\, g^c_a)}{\sqrt{-g}} R^{ab}{}_{di}}$

$\cancel{+ \frac{1}{2} (\partial_a g_{cm}) g^{cm} R^{ab}{}_{di}}$

$(\nabla_c R^{ab}{}_{di}) g^i_b = (\nabla_c R^{ab}{}_{di} g^i_b) - (\nabla_c g^i_b) R^{ab}{}_{di}$

↳ $R^{ba}{}_{id}$

$g^c_a \nabla_c R^a{}_d = g_{ak} g^{kc} \nabla_c R^a{}_d = g_{ak} \nabla^k R^a{}_d$

$= \boxed{\nabla_a R^a{}_d}$

$g^c_a (\nabla_d R^{ab}{}_{ic}) = - g^c_a (\nabla_d R^{ab}{}_{ci}) = -\left[\nabla_d (R^{ab}{}_{ci} g^c_a) - \nabla_d (g^c_a) R^{ab}{}_{ic}\right.$

$= - \nabla_d R^b{}_i$

$g^i_b (-\nabla_d R^b_i) = - \nabla_d (R^b_i g^i_b) + (\nabla_d g^i_b) R^b_i \Rightarrow -\nabla_d R$

$$\therefore \nabla_b R^b_d + \nabla_a R^a_d - \nabla_d R$$

But R is scalar $\therefore \nabla_d R \Rightarrow \partial_d R$

$$2 \nabla_a R^a_d - \nabla_d R = 0$$

$$\nabla_a R^a_d - \frac{1}{2} \nabla_d R = 0$$

$$\Rightarrow \nabla_b R^b_d - \frac{1}{2} \delta^b_d (\nabla_b R) = 0$$

$$\Rightarrow \nabla_b R^b_d - \frac{1}{2} \nabla_b (\delta^b_d R) = 0$$

$$\Rightarrow \nabla_b \left[R^b_d - \frac{\delta^b_d R}{2} \right] = 0$$

But from Th. $\delta^b_d = g^b_d$

$$\Rightarrow \nabla_b \left[\underbrace{R^b_d - \frac{g^b_d R}{2}}_{G^b_d \equiv \text{Eistein Tensor}} \right] = 0$$

④ Divergence w.r.t. any ind. of $G_{ab} = 0$

$$\therefore \nabla_b G^b_d = 0 \Rightarrow \boxed{\nabla^k G_{dk} = 0}$$

Divergence of Einstein Tensor = 0

⑤ Doubt As G^b_d is sym. $\therefore$ Divergence of Eistein Tensor on Both indices = 0

⑥ **Field Equ**

$R^a_{bcd} = \partial_i \Gamma = \partial_i \partial_a \phi \rightarrow$ from Geodesic Deviation Acc.

Doubt

We Can get $\partial_i \partial_a \phi$ by R^a_{bcd} by Contracting and ~~then dyt~~

we want field eqn which will generalize $\partial^2\phi = 4\pi\rho q$. ∴ In GR, field eqn $R^a{}_{bcd}$ should be there.

⑦ $$A = -m\int d\tau - q\int A_i\,dx^i - \frac{1}{16\pi}\int F_{ab}F^{ab}\,d^4x$$

$-m\int d\tau$ ↑ free action for particle

$q\int A_i\,dx^i$ ↑ How charge is coupling to given External A_i field

$\frac{1}{16\pi}\int F_{ab}F^{ab}\,d^4x$ ↓ free action for field — How the field changes given J^k

x^i, t : Ind. var.

A_j, ϕ, x^α : Dependent variable

$$L_f(A_j, \partial_k A_j) \equiv L_f(\phi, \partial_i\phi) \equiv L(\dot{x}^\alpha, u^\alpha)$$

As A is G.I. $L_f(F_{ab}) \longrightarrow$ Due to L.I. $L_f = F^{ab}F_{ab}$ or $\epsilon^{abcd}F_{ab}F_{cd}$ $\longrightarrow A = \frac{1}{16\pi}\int F_{ab}F^{ab}\,d^4x$

$$L_f(\phi, \partial_i\phi) = \frac{\partial_i\phi\,\partial^i\phi}{2} - U(\phi)$$

⑧ Going with flow for Grav.

$$A = -m\int d\tau - q\int (B_i|\phi)\,dx^i \;\cancel{-\frac{1}{16\pi}\int F_{ab}F^{ab}\,d^4x}$$

$$-\int L_{Grav}\,d^4x$$

But we have already seen this leads to wrong result : Light Bending.

Two masses attraction.

⑨ Matter : Anything other than Gravity

$$A_M \overset{\boxed{SR}}{=} \int L_m(\phi, \partial_i \phi)\, d^4x$$

↑ x, A_j ⟶ Ord. Deriv.

$$A_m = \int \frac{F^{ab} F_{ab}}{16\pi} d^4x$$

Suppose there is massive particle around just + EM field

in EM
By ⑦ $$L_m(A_j, \partial_j A_j) \equiv \frac{F^{ab} F_{ab}}{16\pi}$$

⑩ $$A_M \overset{\boxed{GR}}{=} \int L_M(\phi, \nabla_i \phi) \sqrt{-g}\, d^4x \quad \text{— ①}$$

See L-9 ㉜ $\hookrightarrow \equiv L_m(\phi, \partial_i \phi)$ as $\nabla_i \phi = \partial_i \phi$

$\sqrt{g_{ab} dx^a dx^b} = L(x^a, \partial_i x^a)$

Doubt $-m\int ds - q\int A_i dx^i$ are already incorporated in this.

⑪ We can take ① & everything will work out
But
If we want to start with EOM & then generalize.

Doubt

⑫ $\nabla_i F^{ik} = 0$

$$\partial_i F^{ik} = \partial_i(\partial^i A^k - \partial^k A^i)$$
$$= \Box A^k - \partial^k \partial_i A^i = 4\pi J^k$$

Never going to find solu of this Equ so it is stupid to solve for this without imposing Gauge Condition

RHS = 0 in free space.

Using Gauge (Lorentz) $\partial_i A^i = 0$

$\Box A = 0$

⑬ By ⑪

$\Box A^k = \partial_i \partial^i A^k = 0$

Should generalize to

$\Box A^k = \nabla_i \nabla^i A^k = 0$

⑭ But we don't get this

~~$\nabla_a F^{ab}$~~ $\nabla_i F^{ij} = 0$

$\nabla_i \left(\nabla^i A^j - \nabla^j A^i \right) = 0$

$\nabla_i \nabla^i A^j - \nabla_i \left(\nabla^j A^i \right) = 0$

$\Box A^j - \nabla_i \left(\nabla^j A^i \right) = 0$

$\uparrow$ Don't commute like $\partial_i \partial^j$

$(\nabla_i \nabla_j - \nabla_j \nabla_i) A^i = R^i{}_{kij} A^k = R_{kj} A^k$

as this is contraction on 1 & 3

$\therefore R^i{}_{kij} = R_{kj}$

$\Box A^j - \nabla^j \nabla_i A^i - R^j_k A^k = 0$

Using Gauge Lorenz Condn $\nabla_i A^i = 0$

$$\boxed{\Box A^j - R^j_k A^k = 0}$$

$\uparrow$ Too small to be detected

⑮ ∴ Correct approach is to modify action & then obtain EOM.

⑯ $A_m = \int L_m(g^{ab}, \Phi) \sqrt{-g}\, d^4x$

$\cancel{L_m(x^i, \partial_a x^i)}$

$\cancel{\Phi = (\nabla_i \phi)}$ ϕ: generic symbol.

$= \cancel{\partial_i \phi}$

for all Exp. purpose: minimal Coupling works.

⑰ why not?
$$A_m = \int R\, L_m(g^{ab}, \Phi) \sqrt{-g}\, d^4x \quad (+RL)$$

Explicit coupling to Curvature can never be determined by Princ. of Eq.

$\Rightarrow$ In flat spacetime $R = 0$ & get back same Lag.

<u>Principal of minimal coupling</u>

Eg for scalar field $A_m = \int (\partial_a \phi \partial_b \phi R^{ab} + \partial_a \phi \partial^a \phi)$

These things cannot be excluded by Princ. of Eq.

Principle of Eq. is local principle which works in infinite small. region. But here curvature can be there & ∴ Coupling to Curvature can never be determined

⑱ $$A_{total} = \int \overbrace{L_m(g^{ab}, \Phi) \sqrt{-g}\, d^4x}^{L_m} + \int \overbrace{L_{grav}(g^{ab}) \sqrt{-g}\, d^4x}^{L_{grav}}$$

We need L_{grav} for same reason we need ~~EM field~~

just like $L(A_j, \partial_k A_j)$

$\Rightarrow L(g^{ab}, \partial_k g^{ab})$

But there is no way I can construct scalar from this.

(19) Dynamical Variable ~~g~~ ϕ

$$\therefore\ L(\phi, \partial_i \phi) \equiv L(g^{ab}, \partial_i g^{ab})$$

just as $L(A_j, \partial_i A_j) \equiv L(F^{ab})$.

Why I can't construct a Scalar from $g^{ab}, \partial_k g^{ab}$?

$g_{ab} g^{ab} + g_{cd} g_{ab} \partial_k g^{ab} \partial^k g^{cd} \rightarrow$ scalar?

as going to Inertial local frame
$g_{ab} = \eta_{ab}$
$\partial_i g_{ab} = 0$
∴ No ~~Non~~ Trivi

No Non Trivial Scalar can be made.

(20) ∴ 2nd Derivatives of g_{ab} is Req.

$$L(q, \dot{q}, \ddot{q})$$

∴ 3rd order DE will be obtained → which is bad

But for ~~some~~ Reasons we get 2nd order DE

how δg^{ab} came if we are δ w.r.t g_{ab}

$$\overset{\frac{T_{ab}}{2}\delta g^{ab}\sqrt{-g}}{} \qquad \overset{-E_{ab}\sqrt{-g}}{}$$

(21) $$\delta A_{tot} = \int \frac{\delta \mathcal{L}_m}{\delta g^{ab}} \delta g^{ab} d^4x + \int \boxed{\frac{\delta L_{grav}}{\delta g^{ab}}} \delta g^{ab} d^4x = 0$$

w.r.t. g_{ab}

just like we did w.r.t. A_j to obtain field Eq. $\partial_i F^{ik} = 4\pi J^k$

just like J_n is acting External source in EM

Why T_{ab} is the source?

as T_{ab} is coming from matter & as Grav. field is affected by matter ∴ T_{ab} is the source.

$$\Rightarrow \boxed{E_{ab} = \frac{T_{ab}}{2}} \quad \text{EOM}$$

(22) From matter Action we are picking T_{ab} which is acting as source.

as Action has to be L.I. or Scalar.

(23) $$\delta A_m = \frac{1}{2}\int \delta g^{ab}\, T_{ab} \sqrt{-g}\, d^4x$$

(24) Properties of T_{ab}

① T_{ab} is 2 Rank Tensor as $\frac{\delta L_m}{\delta g^{ab}} = T_{ab}$

② T_{ab} is Sym as it is obtained from Action $\int T_{ab}\,\delta g^{ab}$

(25) in SR

$\partial_a T^a_{\ b} = 0 \Rightarrow \nabla_a T^a_{\ b} = 0$

Now we want to prove $\nabla_a T^a_{\ b} = 0$ though we don't know what $T^a_{\ b}$ is from (24)

(26) Under $x^a \to x^a + \xi^a$

$$\delta g^{ab} = \nabla^a \xi^b + \nabla^b \xi^a$$

(27) $\delta A_m = \frac{1}{2}\int \sqrt{-g}\, d^4x\, T_{ab}\left(\nabla^a \xi^b + \nabla^b \xi^a\right) + \delta A_1$

Net Change $\delta A_m = \left(\frac{\delta A_m}{\delta \phi}\right)_g \delta\phi + \left(\frac{\delta A_m}{\delta g^{ab}}\right)_\phi \delta g^{ab}$

as T_{ab} is sym

$$= \int \sqrt{-g}\, d^4x\, T_{ab} \nabla^a \xi^b$$

$$= \int_V \sqrt{-g}\, d^4x\, \nabla_a\left(T^a_{\ b}\, \xi^b\right) - \int_V \sqrt{-g}\, d^4x\, \xi^b \nabla_a T^a_{\ b} + \delta A_1$$

We were already changing about g_{ab} Now x^i? How?

under $x^a \to x^a + \xi^a$

(28) $\delta(L\sqrt{-g}) = L\left(-\frac{g_{ab}}{2}\delta g^{ab}\sqrt{-g}\right) + \sqrt{-g}\,\delta L$

$$= L\left(-\frac{g_{ab}}{2}\left(\nabla^a\xi^b + \nabla^b\xi^a\right)\sqrt{-g}\right) + \sqrt{-g}\,\delta L$$

δg^{ab}

$$= L\left(-g_{ab}\nabla^a\xi^b\sqrt{-g}\right) + \sqrt{-g}\,\delta L$$

as in (23) $L \to 0$

$$= -L\, \nabla_a \xi^a \sqrt{-g} + \sqrt{-g}\,\delta L$$

$$= -\sqrt{-g}\left(L\nabla_a\xi^a + \xi^a\nabla_a L\right) = -\sqrt{-g}\,\nabla_a\left(L\xi^a\right)$$

(29) $\delta L = \bar{L}(x) - L(x)$

change in functional form

$$= \bar{L}(\bar{x} - \xi) - L(x)$$

$$= \bar{L}(\bar{x}) - \xi^a \underset{\overset{||}{\nabla_a L}}{\nabla_a L'} - L(x)$$

$\nabla_a L' = \nabla_a \overset{0}{\delta L} + \nabla_a L$

L is scalar $\therefore L(x) = \bar{L}(\bar{x})$

(30) $\delta\left(L\sqrt{-g}\right) = -\sqrt{-g}\,\nabla_a\left(L\xi^a\right)$ why Lag is scalar?

(31) $\int \delta A_m = \int \delta(L\sqrt{-g})\, d^4x$

as $A_m = \int L \sqrt{-g}\, d^4x$

where L is scalar

$\int \delta A_m = -\sqrt{-g} \int \nabla_a (L \xi^a)\, d^4x$

I can always choose this

on surface if ξ^a vanishes. $\delta A_m = 0$

as ξ^a is in my hand ∴ $\delta A_m = 0 \; \forall \xi^a$ ∴ $\nabla_a T^a_b = 0$ is identity.

(32) Using (27)

RHS 1st term vanishes

$\delta A_m = 0$

∴ $\nabla_a T^a_b = 0 \iff$ EOM → how?

(33) In EM

when I make coordinate transfn $x^a \longrightarrow x^a + \xi^a$

then my A_j will change

and as I am varying w.r.t A_j

δA_m will pick up extra term

i.e. $\frac{\delta L_m}{\delta A_j} \delta A_j \Big|_{x^a+\xi^a}$

Doubt?

In Landau

(34) We are not considering this

Considering $\delta A_j = 0$

? as L_m is assumed to be Lorentz invariant

(35) $\nabla_a T^a{}_b = 0 \longleftrightarrow$ EOM

Pg ~~213~~ BOOK

$T^a{}_b$ is of a field

$T^a{}_b$ is of a particle

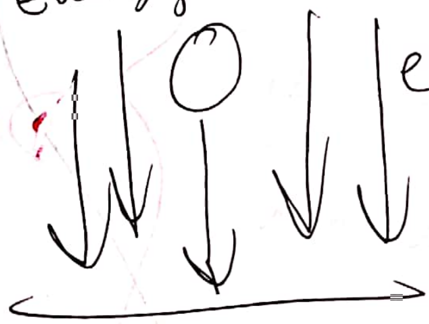
MSR $\partial_a T^a{}_b$ goes to conservation of Both Energy & Momentum.

Conservation law cannot be obtained if $\partial_b g_{ac} \neq 0$

(36) from $\nabla_a T^a{}_b = 0$ from this one can obtain

see L-~~10~~ (9) $\leftarrow \nabla_a (T^a{}_b \xi^b_{(\ell)}) = 0$

∴ Energy Mom. Tensor is not a conserved qty.



Energy of Ball does not Remain Constant.

$\rightarrow \nabla_a T^a{}_b$ is not a Conservation law in Curved Spaceti

earlier $\delta A_{\phi, A^a} = \delta \int d\tau = 0$ (Particle traj fixed or given)

(37) $A_m = -m\int ds = -m\int\sqrt{g_{ab}\, dz^a dz^b}$ $\quad z^a(s)$: Traj

$\delta A_m = -\sum_a m_a \int \frac{1}{2}(\delta g_{ab}) u^a u^b\, ds$, using $\delta g^{ak} = -g^{ab} g^{ck} \delta g_{bc}$

w.r.t. g_{ab}

$= \sum_a \frac{m_a}{2} \int u_a u_b\, \delta g^{ab}\, ds = \frac{1}{2}\int \rho\, u_a u_b \sqrt{-g}\, d^4x\, \delta g^{ab}$

$0 = $ B.C.

compare (23)

(38) $$\rho = \sum_A \int m_A \delta_D(x - z_a)\frac{ds}{\sqrt{-g}}$$ → see L-5 (21)

In GR $$\int f(x)\left(\frac{\delta_D(x-a)}{\sqrt{-g}}\right) d^4x\sqrt{-g}$$ ← correct Def. of Dirac Delta

In SR $$\int f(x)\,\delta_D(x-a)\,d^4x$$

⇒ for material particles

Comparing (1) with (23) & get T^{ab} for collection of particles.

(39) $T^{ab} = \rho u^a u^b$ → Think of it like fluid. ρ fluid flowing with 4 velocity u^a. Suppose I am comoving along this fluid. What is moment. & Energy in that frame?

? ©

$T^{ab} u_b = \boxed{\rho u^a}$ → Momentum per Unit Volume. ~~Density~~ Mom. Tensor

as $u^b u_b = 1$ → This is Energy

This gives ρu^0 = Energy Density, $\rho u^\alpha = \rho \gamma v$ = Mom. flux

(40) $\rho = T_{ab}\, \underline{u}^a\, \underline{u}^b$ → T

This need not be the same velocity as in (39). Both U^a matches in Rest frame

∴ $T^a{}_b$ is Energy Mom. Tensor

Dust ⓑ

ideal fluid (Pressureless & ~~without Density~~)

Radiation is ideal fluid

for fluid with (ρ & P)

(41) $$T^{ab} = (\rho + p)u^a u^b - p g^{ab}$$

$$T^a{}_b = (\rho + p)u^a u_b - p g^a{}_b$$
$$= (\rho + p)u^a u_b - p\,\delta^a{}_b$$

$$= \rho u^a u_b - p\,\underbrace{(\delta^a_b - u^a u_b)}_{\text{Projection Tensor } P^a_b}$$

without Pressure

Projection Tensor P^a_b ↳ (i) Symm Tensor

(42) $P_{ab} = P_{ba}$

$P_{ab} u^a = u_b - u_b = 0$

∴ Projection Tensor $P_{ab} \perp$ u^a

$$\left\{ \begin{array}{l} P_{ab} v^b = v^{\perp}_a \\ v^{\perp}_a u^a = 0 \end{array} \right\}$$

∴ All the vectors ⊥r to u^a can be Obtained

Which is Eq. to. $\vec{v} - \vec{n}(\vec{v}\cdot\vec{n}) = v^{\perp}$ if u^a, u^b are Unit vectors

here n is // to surface

(43) ∴ $T^a_b = \rho u^a u_b - p P^a_b$

lives in orthog. space to U^a as $P_{ab} U^a = 0$

If I go in Rest frame of fluid then $u^a = (t, 0, 0, 0)$

∴ P_{ab} is all space

$T^a_b u^b = \rho u^a$ is mom. flux

∴ $T^a_b = \rho u^a u_b - p P^a_b$

Pressure acts on Space and coeff. of P^a_b is called Pressure

How do I know that T^a_b is energy ~~Densit~~ Mom Tensor?

(44) $T^a{}_b u^b = \rho u^a u_b u^b - \cancel{p\, P^a{}_b u^b} = \rho u^a$

from (43) $T^a{}_b u^b = \rho u^a$ compare with (39)

Pressure forces/Internal stresses do not contribute to bulk energy flow/momentum flow. $T^a{}_b$ is Energy mom. Tensor

Because they cancel out themselves

(45) $T^a{}_b = (p+\rho) u^a u_b - p\, \delta^a{}_b$

Trace $T^a{}_a = (p+\rho) - 4p = \rho - 3p$

For Radiation ≡ ideal fluid $p = \frac{1}{3}\rho$

$\therefore T^a{}_a = 0$

This is true for any EM field

(46) Conformal Transfn

$$g_{ab} \longrightarrow \Omega^2(x)\, g_{ab} \longrightarrow g_{ab} + \underbrace{\epsilon(x)\, g_{ab}}_{\delta g_{ab}}$$

$\Omega^2 \downarrow 1 + \epsilon_e(x)$

lowest order in ϵ

as $g = \begin{vmatrix} g^{00} & - - - \\ g^{10} & - - \\ g^{20} & - - \\ g^{30} & - - \end{vmatrix} \to$ Digonalize $\to \begin{vmatrix} g^{00} & & & \\ & g^{11} & & \\ & & g^{22} & \\ & & & g^{33} \end{vmatrix}$

$\therefore \Omega^8$

(47) EM Action Remain Invariant under conf. Transfn

in 8 Dim - Not conf. Inv

(48) $A_{EM} = -\frac{1}{16\pi}\int d^4x \sqrt{-g}\, F_{ab} F_{em}\, g^{ae} g^{bm}$

$\sqrt{-g} \downarrow \Omega^4 \sqrt{-g}$, $g^{ae} \downarrow \frac{1}{\Omega^2} g^{ae}$, $g^{bm} \to \frac{1}{\Omega^2} g^{bm}$

(49) Th - $g^{ab} \longrightarrow \frac{1}{\Omega^2} g^{ab} \longrightarrow$ general

$\therefore \bar{A}_{em} = A_{em}$

$\sqrt{-g} \longrightarrow \Omega^4 \sqrt{g}$

Action is conf. Invariant

(50)

Depends on Dimension

(50) Proof : $\boxed{g_{ab} \longrightarrow \Omega^2 g_{ab}.}$

$g_{ab} = g_{ak} g_{bl} g^{kl} \longrightarrow \Omega^2 g_{ak} \Omega^2 g_{bl} K g^{kl}$ — ①

$\Omega^2 g_{ab} = \Omega^2 g_{ak} g_{bl} g^{kl}$ — ②

But ① & ② are Equal

$\therefore c = \frac{1}{\Omega^2}$

$\therefore \boxed{g^{ab} \longrightarrow \frac{1}{\Omega^2} g^{ab}}$ Depends on the Dimension

(51) Proof : $\sqrt{-g} \longrightarrow \Omega^4 \sqrt{-g}$ — 4 comes 4D

~~Assuming~~ g_{ab} ~~is Diag. Matrix~~

$$g = \begin{vmatrix} g_{ab} \end{vmatrix} = \begin{vmatrix} g_{ab} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{vmatrix}$$

$\therefore$ each $g_{ab} \longrightarrow \Omega^2 g_{ab}$

$\therefore g \longrightarrow \Omega^8 g$

Non Diag. Terms will not Contribute.

$\therefore \sqrt{-g} \longrightarrow \Omega^4 \sqrt{-g}$.

(52) for EM $\longrightarrow A = \int L_m(\phi, \nabla\phi, g^{ab}) \sqrt{-g}\, d^4x$

$\delta A_m = \frac{1}{2} \int \sqrt{-g}\, d^4x\, T_{ab}\, \delta g^{ab}$

It Conformally Inv. $\therefore \delta A_m = 0 = -\frac{1}{2} \int \sqrt{-g}\, d^4x\, T^{ab}\, \epsilon(x)\, g_{ab}$

(53) As $\delta g_{ab} = \epsilon g_{ab}$

$\delta(g_{ak} g_{bl} g^{kl}) = \epsilon g_{ab}$

$\epsilon g_{ab} + \epsilon g_{ab} + k g_{ab} = \epsilon g_{ab}$

$k = -\epsilon$ $\hookrightarrow$ Bec. $\delta(g_{ak} g^{k}_{b}) = \delta g_{ab}$

$\quad\quad\quad\quad || $

$(\delta g_{ak}) g^{k}_{b} + (\delta \cancel{g^{k}_{b}}) g_{ak}$

$\therefore \underline{\underline{\delta g^{ab} = -\epsilon g^{ab}}}$

(54) $\delta A_m = -\frac{1}{2} \int \sqrt{-g}\, d^4x\, T^{ab} g_{ab}\, \epsilon(x) = 0$ $\forall \epsilon$

$\therefore T^{ab} g_{ab} = T^{a}{}_{a} = 0$

$$\boxed{\therefore \text{for any Action which is Conformally Invariant } T^{a}{}_{a} = 0}$$

(55) EM in 8 Dim.

$\sqrt{-g} \longrightarrow \Omega^8 \sqrt{-g}$

$\therefore$ in 8 Dim. Action will not be conform Inv.

$\therefore T^{a}{}_{a}$ in 8 Dim will not vary.

6) $A = \frac{1}{16\pi} \int F_{ab} F_{em}\, \overset{\nearrow \Omega^2}{g^{ae}}\, \overset{\nearrow \Omega^{-2}}{g^{bm}}\, \overset{\nearrow \Omega^4}{\sqrt{-g}}\, d^4x \longrightarrow$ Inv.

$= \frac{1}{16\pi} \int F^{ab} F^{em}\, \underset{\Omega^2}{\underset{\downarrow}{g_{ae}}}\, \underset{\Omega^2}{\underset{\downarrow}{g_{bm}}}\, \underset{\Omega^4 \sqrt{-g}}{\underset{\downarrow}{\sqrt{-g}}}\, d^4x$ $\rightarrow$ if done in lower index $\longrightarrow \Omega^8 \rightarrow$ Not Inv.

2 – form ?

L-13

① T^{ab} for Material Particles

$$T^{ab} = \rho U^a U^b = \text{Energy Mom. Tensor.}$$

$$T^{ab} U_b = \rho U^a = \text{Momentum Density}$$

② T^{ab} for EM field.

Anything except gravity is matter.

$$A_m = -\frac{1}{16\pi} \int d^4x \sqrt{-g}\, g^{ak} g^{bj} F_{kj} F_{ab}$$

$$\delta A_m = \frac{-1}{16\pi} \int d^4x\, F_{kj} F_{ab} \left[\sqrt{-g}\,(\delta g^{ak})\, g^{bj} + \sqrt{-g}\,(\delta g^{bj})\, g^{ak} + (\delta \sqrt{-g})\, g^{ak} g^{bj} \right]$$

↙ w.r.t. g_{ab}

In SR varied w.r.t. A_j

$$= \frac{-1}{16\pi} \int d^4x\, F_{kj} F_{ab} \left[2\sqrt{-g}\, \delta g^{ak} g^{bj} \quad \frac{\delta g^{ak} g^{bj}}{2\sqrt{g}}\, \delta g \right]$$

$$= \frac{-1}{16\pi} \int d^4x\, F_{kj} F_{ab} \left[2\sqrt{-g}\, \delta g^{ak} g^{bj} - \frac{\delta g^{ak} g^{bj}\, g\, \delta g^{mn} g_{mn}}{2\sqrt{-g}}\right.$$

$$= \frac{-1}{16\pi}\int d^4x\, F_{kj} F_{ab}\left[2\sqrt{-g}\,\delta g^{ak} g^{bj} - g^{ak} g^{bj} \frac{g_{mn}\,\delta g^{mn}\sqrt{-g}}{2}\right]$$

$$\boxed{\sqrt{ab} = \sqrt{-a}\,\sqrt{-B}}\ ?$$

$$\delta A_m = \frac{-1}{16\pi}\int \sqrt{-g}\left[2 F_k^{\ b} F_{ab}\,\delta g^{ak} - \frac{(F^{ab}F_{ab})\, g_{mn}\,\delta g^{mn}}{2}\right.$$

let $F^{ab}F_{ab} = F^2$

$$= \frac{-1}{16\pi}\int \sqrt{-g}\left[2 F_k^{\ b} F_{ab}\,\delta g^{ak} - \frac{F^2 g_{ak}\,\delta g^{ak}}{2}\right] d^4x$$

$$= \frac{-1}{8\pi}\int \sqrt{-g}\,\delta g^{ak} d^4x\left[F_k^{\ b} F_{ab} - \frac{F^2 g_{ak}}{8}\right]$$

$$\boxed{\delta A_m = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\delta g^{ab}\, T_{ab}}$$

③ $$T_{ak} = -\frac{1}{4\pi}\left[F_k^{\ b} F_{ab} - \frac{F^2 g_{ak}}{4}\right]$$

$$T^{ak}_{\ ak} = -\frac{1}{4\pi}\left[F^{kb} F_{ab} - \frac{F^2\delta^k_a}{4}\right] \rightarrow -\text{sign}$$

$$= \frac{1}{4\pi}\left[F^{kb} F_{ba} + \frac{F^2\delta^k_a}{4}\right]$$

④ $$T^a_{\ a} = \frac{1}{4\pi}\left[F^{ab} F_{ba} + \frac{F^{ab} F_{ab}\, 4}{4}\right]$$

$$= \frac{1}{4\pi}\left[F^{ab}_{a} F_{ba} - F^{ab} F_{ba}\right]$$

$= 0$ as EM Action is Conf. Invariant

⑤ Varying Action w.r.t. g_{ab} we get T_{ab}

Varying Action w.r.t. traj. particle we get geod-Eqn

Vary. Action w.r.t. A_i we get Maxwell Eqn in Curved Spacetime.

Can we get Tab from Action in SR?

→ There is a procedure to get ~~Actio~~ Tab but there are 2 problems in this ① Tab not sym ② There is a procedure to make it sym. but it is not uniq

Th: If T^{ab} not sym Angular mom' not conserved

$$⑥ \quad T^{k}_{a} = \frac{1}{4\pi}\left[F^{kb} F_{ba} + \frac{F^2 \delta^k_a}{4} \right]$$

Also valid in SR just $g_{ab} \longrightarrow \eta_{ab}$.

∴ in SR there is no Unique def. Tab

∴ Extra features had to be brought to do that

But in GR & assumed minimal coupling there is Unique.

Why T^{k}_{a} is the Energy Momentum Tensor?

$$⑦ \quad T^{0}_{0} = \frac{1}{4\pi}\left[F^{0\alpha} F_{\alpha 0} + \frac{2(B^2 - E^2)}{4} \right]$$

$$= \frac{1}{4\pi}\left[E^2 + \frac{B^2 - E^2}{2} \right]$$

$$= \frac{1}{8\pi}\left[E^2 + B^2 \right] = \text{Energy Density in SR}$$

↓ EM Energy Density

energy Density true

$$⑧ \quad T^{\alpha}_{0} = \frac{1}{4\pi}\left[F^{\alpha b} F_{b0} \right] = \frac{1}{4\pi}\left[F^{\alpha\beta} F_{\beta 0} \right]$$

↓ work out

$$T^{0\alpha} = \frac{1}{4\pi} (\vec{E} \times \vec{B})^{\alpha} = \text{EM Momentum Density}$$

∴ T^{k}_{a} which is the energy Momentum Tensor of EM field act as a source of Grav. field.

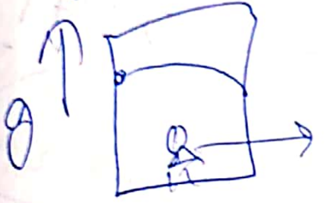
⑨ $T^{\alpha\beta}$ do not have direct interpretation in EM.

But in plasma, they can be thought of as some stresses in magnetic field.

How?

⑩ T^{ik} is the source of gravity? → EM field produce grav. field around it.

If EM radiation is there in a box.



Box went on a curved line

By principle of Eq. light has to bend in presence of gravity

⇓

Grav. field should attract light

light should attract grav. field ⇐

⑪ Scalar field

$$A_{scalar} \overset{SR}{=} \int d^4x \left(\frac{1}{2}(\partial_a\phi\,\partial^a\phi) - V(\phi)\right)$$

$$\overset{GR}{=} \int d^4x\,\sqrt{-g}\,\underbrace{\left[\frac{1}{2}(\partial_a\phi\,\partial_b\phi\,g^{ab}) - V(\phi)\right]}_{L}$$

$$\underset{\text{w.r.t. } g_{ab}}{\delta A_s} = \int d^4x\left[L\left(-\frac{1}{2}g_{ik}\delta g^{ik}\sqrt{-g}\right) + \sqrt{-g}\,\frac{\delta L}{\delta g_{ab}}\,\delta g^{ab}\right]$$

$$= \int d^4x\left[-\frac{L}{2}g_{ik}\,\delta g^{ik}\sqrt{-g} + \sqrt{-g}\,\frac{1}{2}\partial_i\phi\,\partial_k\phi\,\delta g^{ik}\right]$$

$$= \int d^4x\,\frac{\delta g^{ik}\sqrt{-g}}{2}\left[-L g_{ik} + \partial_i\phi\,\partial_k\phi\right]$$

$$\delta A_s = \int \frac{1}{2}\,d^4x\,\sqrt{-g}\,T_{ik}\,\delta g^{ik} \quad \therefore\ T_{ik} = -L g_{ik} + \partial_i\phi\,\partial_k\phi$$

⑫ $T^i_k = \partial^i\phi\,\partial_k\phi - L\delta^i_k$

$H = p\dot{q} - L$

Earlier we made T^i_k from this H.

⑬ $T^0_0 = \dot{\phi}^2 - \left(\frac{\dot{\phi}^2}{2} - \frac{|\nabla\phi|^2}{2} - V\right)$

$= \frac{1}{2}\left(\dot{\phi}^2 + |\nabla\phi|^2\right) + V \equiv H = \frac{\dot{q}^2}{2} + V$

for some suitable V, $T^0_0 > 0$ → Why?

⑭ $L \longrightarrow L - k$ ↗ const. ⟶ EOM remains invariant

$T^a_b \longrightarrow T^a_b + \underbrace{k\delta^a_b}$ from ⑫

Energy Mom. Tensor contributed by constant added to Lag.

5) This concept doesn't ∃ in SR. Coz. adding const. in SR, EOM Remains Same.

How to know this $k\delta^a_b$ is Energy Mom. Tens

No Kinetic Energy Term.

6) Let $L = V + k$
↑ const

$A_{scalar} \longrightarrow A_{scalar} + \int d^4x\sqrt{-g}\,k$

↑ $\sqrt{-g}$ coupled to constant

↗ of any field (254) (6.69)

7) $T^a_b = k\delta^a_b = \text{diag}(k,k,k,k)$

$= \begin{pmatrix} k & & \\ & k & \end{pmatrix}$

Similar happens in T^α_β ~~Symmetric~~ Tensor, & Dist function

⑱ $T^a{}_b = (\rho + P) u^a u_b - p\delta^a_b$ (ideal fluid)

in Rest frame.

$u^a = (1, 0, 0, 0)$

Res $\boxed{T^a{}_b = (\rho + \cancel{p)u^a u}_b - p\delta^a_b = (\rho, -p, -p, -p)}$

⑲ ∴ ⑰ is eq to ⑱ with $p = -\kappa$

⑳ Dark Energy behaves as if it have -ve pressure. which is what is happening here.

㉑ As $T^a{}_b$ has to be +ve.

∴ κ +ve in ⑰

But then p pressure is -ve.

Bec. w.r.t. $\delta_x A$ we assume g is fixed & we know the external field

㉒ when you just add const. to matter lag. Gravity picks up the term which behaves like fluid with -ve pressure

[Matter EOM remains invariant i.e. $\delta_x A = 0$]

why?

?

㉓ Gravity fixes the zero point of the energy. Matter EOM remains inv. if I added constant ∴ By looking at matter I can't tell if const. has been added

→ why? as $\int \kappa\sqrt{-g}\,d^4x \neq 0$? is it x

But by looking at grav. field I can ~~know if that const. is~~ there or not.

$\kappa = 0$ is valid ∵ This energy momentum tensor of vaccum. if QFT field is L.I. then to vaccum fluctuat $T^a{}_b = \kappa\delta^a_b$

㉔ $T^a{}_b = \kappa\delta^a_b$ is the most general Lorentz invariant Energy Momentum tensor. As T_{ab} is 2nd Rank & Sym. ∴ Only available option is ~~g_{ab}~~ δ^a_b

∴ $T^a{}_b = \kappa\delta^a_b$ κ has to be constant so as $T^a{}_b$ is L.I.

Ordinary Derivative

Covariant Derivative of metric = 0

(25) $A_{tot} = A_m(\phi_A, g) + A_g$

$$A_g = \int d^4x \sqrt{-g}\, L_{grav}(g_{ab}, \overset{\uparrow}{\partial_c} g_{ab})$$

(26) Varying w.r.t. g_{ab} A_m would give T_{ab}

A_{field} would give sth.

This method works for everything except gravity.

(27) For field Eqn to be covariant we want our Action to be generally covariant.

But the Converse is not true.

i.e. if the field eqn has to be covariant it is not necessary Action has to generally covariant.

(28) If in Action non covariant part comes as in Total Divergence then field eqn can be generally covariant. Non Cov. Action

∴ Covariant field eqn can be obtained from

(29) But most generally, Action should be gen. Cov

$L_{grav}(g_{ab}, \partial_c g_{ab})$ can't be a scalar
↳

Let $L_{grav}(g_{ab}, \partial_c g_{ab}) = \partial_i g_{ab}\, \partial_j g_{cd}\, g^{ij}\, g^{ab}\, g^{cd} + g^{ab} g_{ab}$

Then going to L.I.f.

$$\left. \begin{aligned} g^{ab} &= \eta^{ab} \\ \partial_i g_{ab} &= 0 \end{aligned} \right\}$$ ∴ All scalars we would get is trivial. ≡ Const.

(30) So $L_{grav}(g_{ab}, \partial_c g_{ab}, \partial_i \partial_j g_{ab}) \equiv L(q, \dot{q}, \ddot{q})$

(31) $$\delta A = \int dt\, \delta L(q, \dot{q}, \ddot{q})$$

$$= \int dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial \ddot{q}} \delta \ddot{q} \right)$$

$$= \int dt \left(\frac{\partial L}{\partial q} \delta q + \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right) \delta \dot{q} \right) + \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}} \delta \dot{q}\right)$$

$$= \int dt \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right)\right)}_{\text{EOM}} \delta q$$

$$+ \int \frac{d}{dt}\left(\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right) \delta q\right) dt + \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}} \delta \dot{q}\right)$$

if $\delta A = \int dt\, \delta L(q, \dot{q})$

$$\delta A = \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \delta q + \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right)$$

$\downarrow$

Only v, q at initial position is req. to evolve the EOM

(32) $$\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right) \delta q \Bigg|_{t=t_1}^{t=t_2} = 0$$

$$\left(\frac{\partial L}{\partial \ddot{q}} \delta \dot{q}\right)\Bigg|_{t=t_1}^{t=t_2} = 0$$

(33) **EOM**: $$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right) = 0$$

3rd Order

in Newton limit 3rd order has to vanish

But how? ∴ 2nd order EOM should be the

(34) 2^{nd} order EOM can be Obtained in $L_{grav}(g_{ab}, \partial_c g_{ab}, \partial_i \partial_j g_{ab})$ case only when $\partial_i \partial_j g_{ab}$ is linear in L_{grav}.

(35) let

$$L_{grav} = L_1(g_{ab}, \partial_c g_{ab}) + \frac{1}{\sqrt{-g}} \partial_k \underbrace{\left(\sqrt{-g}\, Q^k(g, \partial_j g^{ab})\right)}$$

$\uparrow$ 2^{nd} order & will vanish. ∴ we don't have to vary this

But Earlier what we used to do: we vary first & then $\partial_k(\quad)$ comes like $\int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right)$ which we vanish by assuming q at end pts fixed.

(36) ∴ Now will vary L_1 only & get 2^{nd} order EOM

(37) $L_1 = L + \frac{d}{dt} f(q, t)$

$$\int_{t_1}^{t_2} L_1 dt = \int L\, dt + \int dt \frac{d}{dt} f(q,t)$$

$$= \int L\, dt + \int d f(q,t)$$

$$\int L_1 dt = \int L dt + f(q_2, t_2) - f(q_1, t_1)$$

We have assumed q to be fixed at endpts.

$$\therefore \delta \int L_1\, dt = \delta \int L\, dt. \Rightarrow \delta A_1 = \delta A.$$

(38) $L_1 = L + \frac{d}{dt} f(q, \dot{q}, t)$

$S_1 = S + \int d f(q, \dot{q}, t)$ If both $\dot{q}$, q are held fix at end pts then f will not Contribute

only q is fixed at End pts

$\therefore \int d f(q, \dot{q}, t) = f(q_2, \dot{q}_2, t) - f(q_1, \dot{q}_1, t)$

$\delta S_1 = \delta S + \delta f(q_2, \dot{q}_2, t) - \delta f(q_1, \dot{q}_1, t)$

(39) Here we Integerating B/w 2 Hyper surface

$\int L_{grav} \sqrt{-g}\, d^4x = \int L_1 \sqrt{-g}\, d^4x + \int \partial_k (\sqrt{-g}\, Q^k(g, \partial_j g^{ab}))\, d^4x$

If I fix field on Hyper Surfaces then derivative of those fields also get fixed

$$\int \sqrt{-g}\, Q^0(g, \partial_j g^{ab})\, d^3x$$

Derivatives normal to it are not fixed

Derivatives which cannot be Integerated

$\therefore$ There are some terms which can vanish away.

(40) $\therefore$ Now lets choose ~~$\mathcal{L}(g_{ab}, \partial_c g_{ab})$~~

$R^i_{jkl} \Rightarrow R_{jl} = g^k_i R^i_{jkl} = \delta^k_i R^i_{jkl} = R^i_{jil}$

Linear in 1st Deriv. Γ

$\therefore$ Linear in 2nd Deriv. g

$R = g^{ab} R_{ab}.$

$\therefore$ R is linear in 2nd Der. Γ

~~$\frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g}\, Q^k(g, \partial_j g^{ab}))$~~

(41) As we have varied A_j to get field eq.

Here vary w.r.t g_{ab}.

(42) $$A_g = \frac{-1}{16\pi K}\int R\sqrt{-g}\, d^4x$$

$$R = g^{ij}R_{ij}$$

$$\delta(R\sqrt{-g}) = \delta(\sqrt{-g}\; g^{ab}R_{ab})$$

$$= -\frac{1}{2}\sqrt{-g}\, g_{ab}\,\delta g^{ab} R + \sqrt{-g}\,\delta g^{ab}R_{ab}$$

$$+ \sqrt{-g}\, g^{ab}\,\delta R_{ab}$$

$$G_{ab} \longleftarrow \quad = \sqrt{-g}\,\delta g^{ab}\left[R_{ab} - \frac{g_{ab}}{2}R\right]$$

$$+ \sqrt{-g}\,\underbrace{g^{ab}\,\delta R_{ab}}$$

This will turn out to be Total Divergen

(43) $$g^{ab}\,\delta R_{ab}\sqrt{-g} = \sqrt{-g}\, g^{ab}\,\delta(R^{i}_{\ aib})$$

as this is the tensor

∴ I can compute it many frame.

∴ in L.I.f. ⇒ $R^a_{\ bcd} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$

$$\delta R^a_{\ bcd} = \delta(\partial\Gamma) - \delta(\partial\Gamma) + \Gamma\delta\Gamma - \Gamma\delta\Gamma$$

in L.I.f. Γ vanish

$$\therefore \quad = \sqrt{-g}\, g^{ab}\,\delta(\partial_i\Gamma^i_{ab} - \partial_b\Gamma^i_{ai})$$

$$\eta^{ab} \quad 1 \quad = \sqrt{-g}\, g^{ab}\,(\partial_i(\delta\Gamma^i_{ab}) - \partial_b(\delta\Gamma^i_{ai}))$$

$$= \partial_k \{ \sqrt{-g} (g^{ab} \delta \Gamma^k_{ab} - g^{ak} \delta \Gamma^i_{ai}) \}$$

$$= 2 \partial_c (\sqrt{-g}\, g^{bk} Q^{cd}_{ak} \delta \Gamma^a_{bd})$$

(44) Γ^a_{bc} are not Tensors

Th: But $\delta \Gamma^a_{bc}$ are tensors → Scalar ∴ valid in all frames

in ∥ Transport

$$\frac{dv^i}{d\lambda} = - \Gamma^i_{k\ell} \frac{dx^k}{d\lambda} v^\ell \quad \text{——} ①$$

if $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$

then Γ also changes

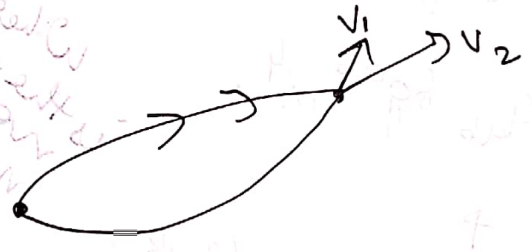
But all other things remain same in ①

$$\therefore \delta v^i_{\parallel} = - \delta \Gamma^i_{k\ell} \dot{x}$$

$$\text{i.e.} \quad \frac{dv^i_1}{d\lambda} = - \Gamma^i_{k\ell} \frac{dx^k_1}{d\lambda} v^\ell_1$$

See Emil's Definition of ∥ Transport

$$\frac{dv^i_2}{d\lambda} = - (\Gamma^i_{k\ell} + \delta \Gamma^i_{k\ell}) \frac{dx^k_2}{d\lambda} v^\ell_2$$



v_1 = Taking vector when g_{ab} is there

v_2 = Taking vector when $g_{ab} + \delta g_{ab}$ is there

Diff. B/W 2 vectors at same pt.

∴ Vector

Now taking the diff.

$$\delta v^2_{\parallel} = - \delta \Gamma^i_{k\ell} \dot{x} v_1$$

is vector

∴ $\delta \Gamma$ is Tensor

(41) A:

(45) Transf. of Γ'

$$\Gamma' = \Gamma L L L + L \partial^2 L$$

↑
This is 2nd- of Γ
in $\delta\Gamma'$
∴ This cancels out

(42)

∴ Transf. of $\delta\Gamma'$ is Tensorial.

Scalar ∴ valid in all frames

(46) from (43)

$$g^{ab}\,\delta R_{ab}\sqrt{-g} = \partial_k\left\{\sqrt{-g}\left(g^{ab}\,\delta\Gamma^k_{ab} - g^{ak}\,\delta\Gamma^i_{ai}\right)\right\}$$

Let $\delta v^k \equiv \underbrace{\left(g^{ab}\,\delta\Gamma^k_{ab} - g^{ak}\,\delta\Gamma^i_{ai}\right)}_{\text{vector}}$ ∴ δv^k is vector

$$\therefore \nabla_k(\delta v^k) = \frac{\partial_k(\sqrt{-g}\,\delta v^k)}{\sqrt{-g}}$$

$$\therefore g^{ab}\,\delta R_{ab}\sqrt{-g} = \sqrt{-g}\,\nabla_k(\delta v^k)$$

(47)

$$(-16\pi k)\,\delta A_g = \int d^4x\,\delta(\sqrt{-g}R)$$

$$= \int_v \sqrt{-g}\,G_{ab}\,\delta g^{ab}\,d^4x$$

$$+ \int_v \sqrt{-g}\,d^4x\,\nabla_k(\delta v^k)$$

$$= \int_v \sqrt{-g}\,G_{ab}\,\delta g^{ab}\,d^4x$$

$$+ \int_{\partial v} |h|^{1/2}\,d^3\cancel{x}\,n_k(\delta v^k)$$

Where n_k is the unit normal $n_k n^k =$ where $\epsilon = \pm$ corre space or time boun

(48) $\delta v^k = g^{ab}\,\delta\Gamma^k_{ab} - g^{ak}\,\delta\Gamma^i_{ai}$

$$\delta\Gamma^k_{ab} = \delta\left(\frac{g^{kc}}{2}\left(-\partial_c g_{ab} + \partial_a g_{bc} + \partial_b g_{ca}\right)\right)$$

Assuming $\delta g^{ab}\big|_{\partial v} = 0$

$$\delta\Gamma^k_{ab} = \frac{g^{kc}}{2}\left(-\delta\partial_c g_{ab} + \delta\partial_a g_{bc} + \delta\partial_b g_{ca}\right)$$

$$\delta\Gamma^i_{ai} = \frac{g^{ik}\,\delta\partial_a g_{ik}}{2}$$

(49) $$\delta v^k = g^{ab}\frac{g^{kc}}{2}\left(-\delta\partial_c g_{ab} + \delta\partial_a g_{bc} + \delta\partial_b g_{ca}\right) - \frac{g^{ab}g^{ik}}{2}\,\delta\partial_a g_{ik}$$

$$= \frac{g^{ak}g^{cb}}{2}\left(-\delta\partial_a g_{cb} + \delta\partial_c g_{ba} + \delta\partial_b g_{ac}\right) - \frac{g^{ak}g^{cb}}{2}\left(\partial_a g_{cb}\right)$$

$$= \frac{g^{ak}g^{cb}}{2}\left(-2\delta\partial_a g_{cb} + 2\delta\partial_c g_{ba}\right)$$

$$= g^{ak}g^{cb}\left(\delta\partial_c g_{ba} - \delta\partial_a g_{cb}\right)$$

(50) $$n_k\,\delta v^k = n^a g^{cb}\left(\delta\partial_c g_{ba} - \delta\partial_a g_{cb}\right)$$

(51) $h_{\alpha\beta} = g_{ab}\,e^a_\alpha e^b_\beta$ where $e^a_\alpha = \dfrac{\partial x^a}{\partial y^\alpha}$

as $ds^2 = g_{ab}\,dx^a dx^b = g_{ab}\dfrac{\partial x^a}{\partial y^\alpha}\dfrac{\partial x^b}{\partial y^\beta}dy^\alpha dy^\beta$

$= h_{\alpha\beta}\,dy^\alpha dy^\beta$

$\beta, \alpha = 1, 2, 3$

$b, a = 0, 1, 2, 3$

(52) As n_a is the normal

$$\therefore \; n_a e^a_\alpha = 0$$

Till this for any T/S/N geodesic

(53) $$\therefore \; h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta = (g_{ab} + \epsilon n_a n_b) e^a_\alpha e^b_\beta$$

$$h_{\alpha\beta} = h_{ab} e^a_\alpha e^b_\beta$$

$$h_{ab} = g_{ab} + \epsilon n_a n_b$$

$$h^{ab} = g^{ab} + \epsilon n^a n^b$$

(54) Inverse of it

$$h^{ab} = h^{\alpha\beta} e^a_\alpha e^b_\beta$$

(55) $$n_k \delta v^k = n^a g^{cb} (\delta \partial_c g_{ba} - \delta \partial_a g_{cb})$$

$$= n^a (h^{cb} - \epsilon n^c n^b) \underbrace{(\delta \partial_c g_{ba} - \delta \partial_a g_{cb})}_{\text{Anti in c \& a}}$$

$$= n^a h^{cb} (\delta \partial_c g_{ba} - \delta \partial_a g_{cb})$$

$$- \epsilon \underbrace{n^a n^c n^b}_{\text{Sym c\&a}} \underbrace{(\delta \partial_c g_{ba} - \delta \partial_a g_{cb})}_{\text{Anti in c\&a}}$$

$$= n^a h^{cb} (\delta \partial_c g_{ba} - \delta \partial_a g_{cb})$$

(56) If $\delta g_{ab}\big|_{\partial V} = 0 \Rightarrow$ $\boxed{(\delta \partial_c g_{ab}) e^c_\alpha = 0}$

Tangential Derivatives also get fixed.

(57) But from (54)

$$h^{cb} = h^{\gamma\beta} e^c_\gamma e^b_\beta$$

Putting in (55)

$$n_k \delta v^k = n^a h^{\gamma\beta} e^c_\gamma e^b_\beta (\delta \partial_c g_{ba} - \delta \partial_a g_{cb})$$

$$= -n^a h^{cb} \delta \partial_a$$

(58) ∴

$$A = \frac{-1}{16\pi k}\left(\int \sqrt{-g}\, d^4x\, G_{ik}\,\delta g^{ik} - \int_{\partial V} \underbrace{n^a h^{cb}\,\delta \partial_a g_{cb}}_{\text{Normal Derivatove term}} \sqrt{h}\, d^3y\right)$$

(59) $$S_{GHY} = \frac{-1}{8\pi k}\int_{\partial V} d^3y\, \epsilon |h|^{1/2} k \qquad \boxed{n^a \partial_a}$$

$K_{ab} = \nabla_\alpha n_\beta e^\alpha_a e^\beta_b$ k: Trace of Extrinsic Curvature.

K_{ab} is comp. of Tang. $\nabla_\alpha n_\beta$ & Normal comp of $\nabla_\beta A^\alpha e^\beta_b$

(60) As $\delta g_{ab}\big|_{\partial V} = 0$

& $h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta$

$$\therefore \delta h_{\alpha\beta}\big|_{\partial V} = \delta g_{ab}\big|_{\partial V} \cdot e^a_\alpha e^b_\beta = 0$$

$$\Rightarrow \delta h_{\alpha\beta}\big|_{\partial V} = 0$$ ~~Ind~~ (Induced metric gets fixed)

(61) As the induced metric is fixed on ∂V, ∴ only qty to be varied is K ∴ h fixed $\delta h = 0$

(62) $$\boxed{K = \nabla_a n^a} = g_{ab}\nabla^b n^a = (h_{ab} - \epsilon n_a n_b)\nabla^b n^a$$

$$\boxed{\begin{aligned} K_{ab} &= \nabla_{(\beta} n_{\alpha)} e^\alpha_a e^\beta_b \\ &= h^{ab} e^\alpha_a e^\beta_b \nabla_{(\beta} n_{\alpha)} \\ &= h^{\alpha\beta}\nabla_{(\beta} n_{\alpha)} \end{aligned}}$$

$$= h^{\alpha\beta}\nabla_\beta n_\alpha$$

But $\nabla_a (n^b n_b) = 0$

$$\Rightarrow n_b \nabla_a n^b = 0$$

$$\therefore K = h_{ab}\nabla^b n^a = g_{ad}\, g_{bc}\, h^{cd}\nabla^b n^a$$

$$= h^{cd}\nabla_c n_d = h^{ab}\nabla_b n_a$$

(63) $K = h^{ab}(\partial_b n_a - \Gamma^{i}_{ba} n_i)$

$\delta K = \delta h^{ab}(\partial_b n_a - \Gamma^{i}_{ba} n_i) + h^{ab}(\delta\partial_b n_a - \delta\Gamma^{i}_{ba} n_i)$

Induced metric gets fixed along ∂V $\leftarrow$

$= h^{ab}(\delta\partial_b n_a - \delta\Gamma^{i}_{ba} n_i)$

$- h^{ab}\Gamma^{i}_{ba}\delta n_i$

$$n_\alpha = \frac{\epsilon\,\partial_\alpha\phi}{|\partial_\alpha\phi\,\partial^\alpha\phi|^{1/2}} = \frac{\epsilon\,\partial_\alpha\phi}{|\partial_\alpha\phi\,\partial_\beta\phi\,g^{\alpha\beta}|^{1/2}}$$

$\nearrow$ Involves metric

When varying Our Boundaries are fixed

$\delta\phi = 0$

& as also we have fixed $\delta g_{\alpha\beta}\Big|_{\partial V} = 0$ on Boundary

$n_i = A(x)\partial_i\phi$

$\delta n_i = n_i\,\delta(\ln A)$

Compare with Paddy Note $\rightarrow$ $\therefore$ $\boxed{\delta n_\alpha} = 0$ though it involves metric

$\delta\partial_\beta n_\alpha = 0$

$h^{ab} e^\alpha_a e^\beta_b$

$$\therefore\ \delta K = -h^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} n_\gamma = -\frac{h^{\alpha\beta}}{2} g^{\gamma\mu}(\delta g_{\alpha\mu,\beta} + \delta g_{\beta\mu,\alpha} - \delta g_{\alpha\beta,\mu}) n_\gamma$$

$$= \frac{h^{\alpha\beta}}{2}\delta g_{\alpha\beta,\mu} n^\mu$$

1) $\delta\oint_{\partial V}\epsilon K|h|^{1/2} d^3y = \oint_{\partial V}\epsilon\,\delta K\,|h|^{1/2} d^3y$

1) How the gravity modifies spacetime?

2) $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$

3) How to know if the spacetime is curved/ flat?

4) Proof that arbitrary $g_{\alpha\beta}$ can't be converted to $\eta_{\alpha\beta}$.

5) Proof that we can always transform to L.I.f?

6) Defn of vector (Abstract & Transformation law)?

Defn of Dual vector (Abstract & Transfn Law?

↓

If metric is defined; both defn become equal.

7) Defn of forms.

8) Defn of Tensor.

9) Multiplication/ Addn / Subtn / Contraction of 2 tensors not defined on manifold?

↓

Integration of vectors/ Tensors not defined?

↓

$\int \partial_i A^{ij} d^4x$ × Not defined. } Any index left in under $\int$ is not defined.

$\int A^i d^4x$ × Not defined }

dA^α is not a Tensor qty for Abstract & transfn law both

11) Definition DA^α, $\nabla_\beta A^\alpha$ ⟶ In terms of Gauge theory

↓ ↳ How A^α is changing in the direction of Tangent vector?

Relation of them with straight line $\frac{Du^\alpha}{d\lambda} = 0$?

$+\Gamma^\alpha_{\beta\gamma} u^\beta A^\gamma = 0$ ↓

$\frac{dv}{d\lambda} = \frac{Dv^k}{d\lambda} e_k \rightarrow \frac{dV}{d\lambda} = \partial_j v^i v^j e_i + v^i v^j \partial_j e_i$

$\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} A^\gamma = 0$ ⟵ $\partial_j e_i = C^k_{ij} e_k$

$A^\alpha \in$ internal space ?

↓

Definition of ∥ Transport ?

$\Gamma v \rightarrow$ Matrix multipn

1st order DE → Always solvable

Transfn law for Γ

→ d for scalar fn; D follows product rule

To prove if D follows prod Rule ⟺ ∇ follows prod. rule.

↓

∇ duals & ∇ Tensors? ⟶ $\nabla_i g_{\alpha\beta} = 0$

(14) Demanding Γ is symm ⟹ Γ in terms of $g_{\alpha\beta}$

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depending on T/S.

⑮ $dl = \sqrt{\pm g_{\alpha\beta} dx^\alpha dx^\beta}$ depending on T/S.

$dl^2 = d\tau^2$ or ds^2 depending on T/S.

$dA = -m\, dl$

⑯ $dl = \sqrt{\pm g_{\alpha\beta} dx^\alpha dx^\beta}$

variations

Euler Lag.

T

S

$\boxed{\text{Metric Ind of } x^i \equiv p^i_{cons}}$

$U^j \nabla_j U^i = k(\lambda) U^i$

$k(\lambda) = \frac{d \ln l}{d\lambda}$

$\frac{du_i}{d\tau/s} = \frac{\partial_i g_{\alpha\beta} U^\alpha U^\beta}{2}$

free particle
Non acc. ⟵ $U^j \nabla_j U^i = \frac{f''}{f'} U^i$

$\frac{du^i}{d\tau/s} + \Gamma^i_{jk} U^j U^k = 0 \longrightarrow U^j \nabla_j U^i = 0 \rightarrow \tau = f(\lambda)$

→ Extremizes means max. proper time for time like Geod.
max. proper distance for spacelike Geod

Specifically,
This is max. for short seperation
& for larger seperation this becomes a saddle point

⇒ In CM, this is minimum for shorter seperation
& for lareger seperation this becomes saddle point.

⑰ What is the weak field limit of EOM for particle?

⑱ What are the uses of Affine parameters?
↓
→ FRW for non null cases
→ FRW for Null Cases } easy to solve

⑲ Why For affinely parameterized Geod. if the Geod. is T/S/N then it remains so?

⑳ $L = \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$

$L_2 = g_{ij} U^i U^j$ } Both gives same EOM

㉑ Geod Eqⁿ ① Extremizes length $\Rightarrow$ ∴ Req. metric
② Tangent vectors $\parallel \Rightarrow$ ∴ Req. Γ

㉒ What are the all formulas of g

$$\swarrow \qquad \searrow$$

Divergence $\qquad \partial_i g$

㉓ Why all local things should match in GR & SR eg. u^i, n_i?

㉔ EM in Curved spacetime

$F_{ik} = \nabla_i A_k - \nabla_k A_i$

$\nabla_i F^{ik} = 4\pi J^k = \dfrac{\partial_i(\sqrt{-g}F^{ik})}{\sqrt{-g}}$

<u>In SR</u> $\quad \partial_i F^{ik} = 4\pi J^k \Rightarrow \partial_k \partial_i F^{ik} = \partial_k J^k = 0$

$\int d^4x\, \partial_k J^k = \int d^3x\, J^0 = \text{const}$

<u>In GR</u>

$$\partial_k \frac{\partial_i(\sqrt{-g}F^{ik})}{\sqrt{-g}} = 0 = \frac{\partial_k(\sqrt{-g}J^k)}{\sqrt{-g}} = \nabla_k J^k$$

$$\int \nabla_k J^k\, d^4x \sqrt{-g} = \int |h|^{1/2} J^0 d^3x = \text{const}.$$

㉕. $\parallel$ Transport requires Γ to be defined. Lie transport requires vector field to be defined in the spacetime.

㉖ 2 ways to define Lie Transf $= \alpha_t v = \text{lt} \dfrac{v_Q - v'_Q}{d\lambda}$

$\delta A^i = A^i(Q) - A^{i'}(Q)$

$\alpha_t v = \text{lt} \dfrac{v_Q - v_{P\to Q}}{d\lambda}$

(27) $\mathcal{L}_t V^i = t^\alpha \partial_\alpha V^i - V^\alpha \partial_\alpha t^i$
$= t^\alpha \nabla_\alpha V^i - V^\alpha \nabla_\alpha t^i$
$=$ Tensor qty.

$\mathcal{L}_t V^i = -\mathcal{L}_V t^i$

(28) Similarly 2 ways to define $\mathcal{L}_t g^{ij}$, $L_t A_i$

① $\mathcal{L}_t g^{ij} = \frac{\cancel{lt}\, g^{ij}(Q) - g^{i'j'}(Q)}{d\lambda}$

② $\mathcal{L}_t g^{ij} = lt \frac{g^{ij}(Q) - g^{i^\circ j^\circ}(\cancel{P})}{d\lambda}_{P \to Q}$

(29) $\mathcal{L}_t A^{ij} = t^\alpha \partial_\alpha A^{ij} - A^{\alpha j} \partial_\alpha t^i - A^{i\alpha} \partial_\alpha t^j$
$= t^\alpha \nabla_\alpha A^{ij} - A^{\alpha j} \nabla_\alpha t^i - A^{ij} \nabla_\alpha t^\alpha$

$\mathcal{L}_t A_i = t^\alpha \partial_\alpha A_i + A_\alpha \nabla_i t^\alpha$
$\Downarrow$
we conclude the $\mathcal{L}$ follows product Rule.

(30) Def. $\mathcal{L}_t f = t^\alpha \partial_\alpha f \equiv \frac{df}{d\lambda}$

(31) $\mathcal{L}_t g_{ij} = \nabla_i t_j + \nabla_j t_i$

$\mathcal{L}_t g^{ij} = -(\nabla^i t^j + \nabla^j t^i)$

(32) Def. of spacetime symmetric $\equiv$ Network of Distances eq
$\Downarrow$
$\mathcal{L}_K g_{ij} = 0$ K vector field $\equiv$ killing field

(33) if $K \in$ Basis vector $\Rightarrow \frac{\partial g_{ij}}{\partial x^i} = 0$

if $\frac{\partial g_{ij}}{\partial x^i} = 0 \Rightarrow K = \overset{\circ}{i} \Rightarrow \mathcal{L}_K g_{ij} = 0$ in general as

if Tensor 0 in one f it remains zero in all other frame

if $K \in$ Killing vector then coord. Syst. can be const. s.t. K is Basis vector in that coord syst & $\therefore \frac{\partial g_{ij}}{\partial x^i} \overset{*}{=} 0$

(34) if K_i is given in one coord system & then a^i_j are found by coord Transf. then K_i can be

Max. of 10 killing fields can be found in sp

$10 = 4 + 3 + 3$
Transl. Boost Rotation

$\Downarrow$ ~~De~~ Sitter Spacetime

Prove

(5) In a symmetric space for an affinely parameterized geodesic $U^i K_i =$ constant along the geodesic & along the cross curves

(6) Just as in lag. mechanics by the symmetries of space conservation eqn comes, similarly in GR by killing fields, conservation qty come.

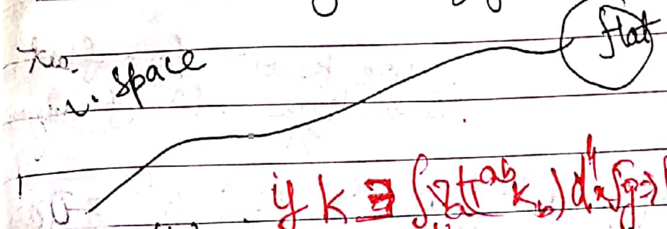
$U^i K_i^{(t)} = \text{const} = \tilde{E} = E/m$; $U^i K_i^{(\phi)} = \text{const} = \tilde{L} = L/m$

7) $\tilde{E} =$ killing Energy $= U^i K_i^{(t)}$

in SR $\partial_a T^a{}_b = 0$

$p^i = \int d^3x\, T^{0i} = \text{const}$

in GR $\nabla_a T^{ab} = 0$



if $K \exists \int_V \nabla_a (T^{ab} k_b) d^4x \sqrt{g} \Rightarrow P^i = \int \sqrt{h}\, d^3x\, T^{0i} k_i = \text{const}$

$K_i^{(t)} U^i = \text{const}$ lets choose coord. where $K_i^{(t)} = (1,0,0,0)$

(51) in that coord. frame $U^0 \overset{*}{=} \text{const}$.

going to flat Spacetime in that coord. frame

$\frac{dt}{d\tau} \overset{*}{=} \gamma \overset{*}{=} \text{const}$.
$\hookrightarrow$ Depends on v

as $U^0 \overset{*}{=}$ constant in that coord. frame it will remain const in any other coord. frame also

$\therefore K_i^{(t)} U^i = \gamma$

$\tilde{E}$: go to ∞ to calculate $\tilde{E}$; can be +/−

(38) $\tilde{E}_{local} = U^i_{obs} U_i^{part.}$; E_{local} is the Energy measured when we go there.
$\hookrightarrow$ Actual ? Precise Def.

Justification: go to the observer frame

$U^{i}_{obser} = (1,0,0,0)$

$\therefore \tilde{E}_{local} = U_0^{particle} = \frac{E_{local}}{m}$

as $\vec{P}_4 = (E, \vec{p})$

9) Prove: For any 2 congruence of curves U^i, k^i

$\mathcal{L}_U k^i = - \mathcal{L}_K U^i = 0$

& if those curves are affinely param. geod.

$\Rightarrow U^i K_i = \text{const}. \Rightarrow U^i K_i = 0$

(40) $\frac{DA^\alpha}{d\lambda} = 0$ is 1st order DE which is always Solvable

∴ It needs one Initial condn.

∴ Along ~~the~~ ~~geod~~. any curve A^α is // Transported

(41) for flat spacetime $\frac{dA^\alpha}{d\lambda} = 0$ ~~coord~~ component Not dep. on co

$\Downarrow$ $\Downarrow$

∴ $k^{+}(\lambda)$ along any curve will be same $\Leftarrow$ Unique Soln

$\Downarrow$

∴ Vector comes back to itself after round.

We know for flat spacetime soln ∃

(42) Condn to give Unique Soln $\partial_i A^\alpha + \Gamma^\alpha_{i\ell} A^\ell = 0$

$\Rightarrow (\partial_m \partial_i - \partial_i \partial_m) A^\alpha = -R^\alpha_{pmi} A^p$

$\Downarrow$

for flat Spacetime $\Rightarrow R^\alpha_{pmi} = 0$

$i, m \equiv$ spacetime index

$\alpha, p \equiv$ Internal spacetime

If the Soln ∃ for $\Gamma \neq 0$ then for that Spacetime $R^\alpha_{pmi} = 0$

(43) Prove The Soln of $\partial_i A^\alpha + \Gamma^\alpha_{i\ell} A^\ell = 0$ doesn't ∃ for $\Gamma \neq 0$ as

$$\Delta V^\alpha = -\frac{R^\alpha_{bcd}}{2} V^b_p \oint v^c dx^d$$

(Proof ?)

(44) $(\nabla_i \nabla_j - \nabla_j \nabla_i) V^k = -R^k_{mij} V^m$

(45) $(\nabla_i \nabla_j - \nabla_j \nabla_i) V_k = R^m_{kij} V_m \Rightarrow R^k_{mij}$ is Tensor

(46) Geodesic Deviation :

(47) Newtonian Limit.

(48) What is the Difference B/W Local If & Box falling freely?

(49) Properties of $R^a{}_{bcd}$ c,d : spacetime Index
a,b : Internal space Index

$R_{abcd} = -R_{bacd}$
$R_{abcd} = -R_{abdc}$
$R_{abcd} = R_{cdab}$
$R_{a[bcd]} = 0$

} Why Properties in L.I.f. will remain true in general?

$\boxed{X = R_{a[bcd]}}$ is A.S. in Any of its indices.

(50) What are the no. Of Ind. components of $R^a{}_{bcd}$?
Why No. of nonzero non vanishing $\partial_i \partial_j g_{ab}$ enters $R^a{}_{bcd}$?

(51) Why 1st & 3rd comp. being contracted?

(52) To prove Ricci Tensor is symmetric?
To prove Einten Tensor is sym?

(53) Bianchi Identity
Prove : $\nabla_{[i} R^a{}_{b[cd]]} = 0$

(54) To prove Divergence of Einsten Tensor = 0.

(55) As $R^\alpha_{\beta\gamma\delta} = \partial_i \Gamma \propto \partial_i \partial^\alpha \phi$
& Newtons Approx $\nabla^2 \phi = 4\pi G \rho$ ∴ $R^\alpha_{\beta\gamma\delta}$ should be there in field eqn

(56) Assumption: All laws can be generalized from flat spacetime to curved.

Eg. $\int ds = \int \sqrt{g_{ij} dx^i dx^j}$ → In acc. frame $g_{ij} = e^{gx}$ → $ds^2 = g_{ij} dx^i dx^j$
$F_{ab} F^{ab}$ → In acc' frame $F_{ab} F^{ab}$ → $F_{ab} F^{ab} \sqrt{-g}$
→ Principle of Equivalence.

(57) The above procedure is not Unique. Explicit coupling to curvature can be there. Eg. $L_m + R$ as $R = 0$ in LIf.
Coupling to curvature can't be determined by Eq. princip.

(58) $A = \int \underset{\substack{\uparrow \\ \text{Matter}}}{L_1} \sqrt{-g}\, d^4x + \int \underset{\substack{\downarrow \\ \text{for gravity field}}}{L_2} \sqrt{-g}\, d^4x$

$\delta A_m = \int T_{ab}\, \delta g^{ab} \sqrt{-g}\, d^4x$ $\delta A_g = \int -E_{ab}\, \delta g^{ab} \sqrt{-g}\, d^4x$

T_{ab} is symmet. Tensor.

(59) T_{ab} is internally there in GR unlike Scalar & EM field.

(60) Prove $\nabla_a T^{ab} = 0 \iff$ EOM

(61) Prove For Many particles $T^{ab} = \rho U^a U^b$ (Dust)

↓

Pressureless

(62) For Ideal fluid

Prove: $T^{ab} = (\rho + p) U^a U^b - p g^{ab}$

Radiation is the ideal fluid.

(63) $T^{a\cancel{b}}_{b} = \rho U^a U_b - p(\underbrace{\delta^a_b - U^a U_b})$

P^a_b = Projection Operator

as

$P^a_b u^b = 0$

64) $T^a_b = \rho U^a U_b - p P^a_b$

in Rest frame $u^b = (1,0,0,0) \Rightarrow P^a_b \in$ Spacelike

∴ pressure acts on spac

5) $T^a_a = (\rho + p) - \delta^a_a p = \rho - 3p$

rove for Ideal fluid $p = \frac{\rho}{3}$ ∴ $T^a_a = 0$

6)

isolate one pt. with other on sphere

$f(\theta) d\theta^2$ why? → similar for $\beta(\theta, \phi)$

$\times \theta, \phi$ $\quad r(r,t)$ $\quad$ introduce dt $\quad$ $\alpha, \mu(r,t)$

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$dt\,d\theta, dt\,d\phi$ ⇓ 0

① $ds^2 = \alpha dt^2 - \beta dr^2 - r(d\Omega^2) - \mu\, dr\, dt$

spherical symmetric - at $r=0$ it should look like surface sphere.

find killing vectors describing spher. symmetry

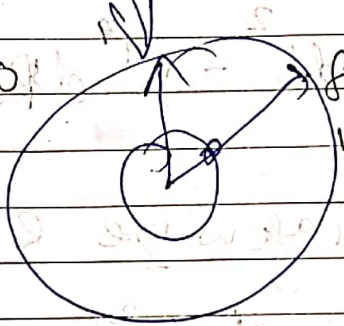
⑦ $g_{\alpha\beta}$ given

↓

$\nabla_{(\alpha} K_{\beta)} = 0$ solve

↓

find K_β



② Here K_β given

↓

find $g_{\alpha\beta}$ which will obey $\nabla_{(\alpha} K_{\beta)} = 0$

Why can't we have $dr\,d\theta$, $dr\,d\phi$

② if $\alpha \neq (r,t)$ $\alpha(\theta, \phi)$ then dynamical eqn depend on which Angular coordinate I am choosing

③ Th: Any quad. form in 2 variable can be diag.
∴ $dr\,dt = 0$

④ ∴ Only By sph. sym:

$$ds^2 = \alpha\, dt^2 - \beta\, dr^2 - r\, d\Omega^2$$

$\alpha, \beta, r(r,t)$

Doubt

$(t, r, \theta, \phi) \to (t', r, \theta, \phi)$ $\quad t \to t'$

$dt' = A\,dt \pm B\,dr$

$(dt')^2 = A^2 dt^2 + B^2 dr^2 \pm 2AB\,dt\,dr$

where $A^2 = \alpha$
$2AB = \mu$
$B^2 - \beta =$

$r \longrightarrow R$ $\quad R^2 = r(r,t)$

~~$R(r,t)$~~ $\quad (t,r,\theta,\phi) \longrightarrow (t,R,\theta,\phi)$

⑤ $ds^2 = \alpha dt^2 - \beta dr^2 - \gamma (d\Omega^2)$

~~$\gamma(r) = r^2$~~ $\quad R^2 = r(r,t)$

$$2dR = \frac{\partial \gamma}{\partial r} dr + \frac{\partial \gamma}{\partial t} dt$$

$$ds^2 = \alpha \alpha' dt^2 - \beta' dR^2 - R^2 d\Omega^2 - f\, dr\, dt$$

Again Diagonalize & let $R = r$

$$ds^2 = \alpha' dt^2 - \beta' dr^2 - r^2 d\Omega^2$$

⑥ $A = \int ds$ $\quad$ or ~~$A = \int$~~

$$= \int \sqrt{\alpha' dt^2 - \dots}$$

$$= \int \frac{\sqrt{\quad}}{dt}\, dt$$

$$= \int \sqrt{\alpha' - \beta' \dot{r}^2 - r^2 \left(\frac{d\Omega}{dt}\right)^2}\, dt$$

$$= \int L\, dt \quad \Big\} \longrightarrow A = \int L^2 d\lambda$$

λ is affine

$$L' = L^2 = \alpha' \left(\frac{dt}{d\lambda}\right)^2 - \beta' \dot{r}^2 \dots$$

$\alpha', \beta'(r,t)$ $\quad$ find Γ, R By this

$$\frac{d}{d\lambda}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{\partial L'}{\partial q}$$

$$q = t \hookrightarrow \frac{d}{d\lambda}\left(\frac{\partial (\alpha' \dot{t}^2)}{\partial \dot{t}}\right) = \frac{\partial L'}{\partial t}.$$

Let Metric be static

$\therefore \alpha', \beta' \times t$.

$\therefore \frac{\partial L'}{\partial t} = 0$

$\therefore \frac{d}{d\lambda}\left(2\alpha'\dot{t}\right) = 0$

$i = 0$

$\Gamma^{t}_{tr} \neq 0$

all other zero

$2\alpha'\ddot{t} + 2\dot{t}\frac{\partial \alpha'}{\partial r}\dot{r} = 0$

$\Rightarrow \boxed{\ddot{t} + \frac{\partial\alpha'/\partial r}{\alpha'}\dot{t}\dot{r} = 0} \longrightarrow \frac{d^2x^i}{d\lambda^2} + \Gamma^{i}_{\mu\nu}U^{\mu}U^{\nu} = 0$

Similarly

$q = r \quad \frac{d}{d\lambda}\left(-\beta'\dot{r}\right) = \frac{\partial}{\partial r}\left(\alpha'\dot{t}^2 - \beta'\dot{r}^2 - r\dot{\theta}^2\right)$

$q = \theta, \phi$

Assuming No staticity $\alpha, \beta\,(r,t)$
~~we got By EFE these 2 eqn~~
& using EFE in vaccum $\Rightarrow G_{ik} = 0$

⑦ $\dot{\beta} = 0$

$\Downarrow$

$g^{\alpha}_{\beta}R^{\beta}_{i\alpha j} = 0 \leftarrow R_{ik} = 0$

$\left.\begin{array}{l} r\frac{\alpha'}{\alpha} + (1-\beta) = 0 \\ r\frac{\beta'}{\beta} - (1-\beta) = 0 \end{array}\right\} \longrightarrow \alpha'\beta + \beta'\alpha = 0$

$\frac{\partial(\alpha\beta)}{\partial r} = 0$

$\alpha\beta = q(t)$

$\alpha(t,r) = \frac{q(t)}{\beta(t,r)} = \frac{q(t)}{\beta(r)}$ (time dep. & spatial dep. seperate it out)

$\dot{\beta} = \frac{\partial\beta(r,t)}{\partial t} = 0 \quad \therefore \beta(r)$

⑧ $ds^2 = \frac{g(t)\,dt^2}{\beta(r)} - \beta(r)\,dr^2 - - - -$

$dt' = \sqrt{g(t)}\,dt$

$\therefore \quad ds^2 = \frac{dt'^2}{\beta(r)} - \beta(r)\,dr^2 - - -$

$\therefore \quad g(t) = 1$

$\therefore \quad \alpha(t, r) = \frac{1}{\beta(r)}$

$\therefore \quad \alpha(t, r) = \alpha(r)$

$\therefore \quad \boxed{\alpha = \frac{1}{\beta}}$

⑨ $\frac{r}{\alpha}\alpha' + \left(1 - \frac{1}{\alpha}\right) = 0$

$r\alpha' + (\alpha - 1) = 0$

$$\int \frac{d\alpha}{\alpha - 1} = -\int \frac{dr}{r} = \ln\left(\frac{c}{r}\right) = \ln(\alpha - 1)$$

$c = r(\alpha - 1)$

$$\boxed{\begin{gathered} \alpha = 1 + \frac{c}{r} \\ \beta = \frac{1}{\alpha} \end{gathered}}$$

(16) $ds^2 = \left(1 + \frac{c}{r}\right)dt^2 - \frac{dr^2}{\left(1+\frac{c}{r}\right)} - r^2 d\Omega^2$

weak far away

$g_{00} = 1 + 2\phi$

$\phi = -\frac{GM}{r}$

$r >> 0$

~~$ds^2 = dt^2 - dr^2$~~

$ds^2 = g_{00}\,dt^2 - \delta_{\alpha\beta}\,dx^\alpha dx^\beta$

in polar coord

$ds^2 = g_{00}\,dt^2 - dr^2 - r^2 d\Omega^2$

$ds^2 = \left(1+\frac{c}{r}\right)dt^2 - \left(1-\frac{c}{r}\right)dr^2 - r^2 d\Omega^2$

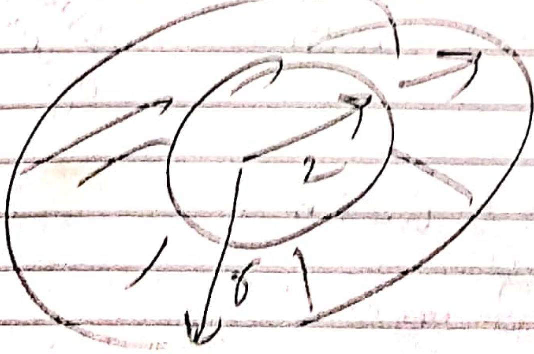
$c = -2GM = r_s$

Assuming we are solving for outside region

Birkhoff theorem

We have not assumed till now that body is static but the region outside would still be static

But for deriving this we assumed $r >> L$



L is radius of body

But if $r = r_s$

& $r_s > L$

then there is problem

But let assume

∴ No problem ⟵ $L > r_s$

Birkhoff's theorem

⑪ Source can be Time Dep. in Sph. Symmetry collapsing in Sph. Symm

$\Downarrow$

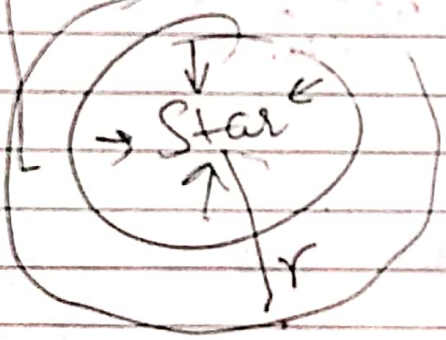
Metric outside it is still static

Any system which follows EFE vaccum & syst is spherically Symm

Equal

Gauss theorem

⑫ Newton



$fdt^2 - f^{-1}dr^2 - r^2 d\Omega^2$

Now assuming this is the metric given to us.

⑬ Shell

Outside same metric with $c = 2GM$

inside same metric But different C

But here $r=0$ metric should hold

~~as EFE = 0 vaccum~~

$\therefore c = 0$

Here at $r=0$ metric is not valid coz we have found it for outside as EFE ≠ 0 vacuum

④ But if c = 0

flat Spacetime

∴ inside shell No grav. field

∴ No Curvature

In Newtonian theory
In shell Sphere
ϕ = Constant inside
In shell Ellipsoid/weird configm
ϕ = const. inside

But in GR only for Spherical Symm. Cases ϕ = const

(16) L-15

Suppose this is the metric given

① $ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2 d\Omega^2$

Put it in EFE

→ Calculate LHS → get T^{ab}

$g_{ab} \rightarrow \Gamma \rightarrow R$

↓ if T^{ab} is this then we have a solution of EFE ~~Eq of motion~~

By EFE we get

② $\boxed{T^0_0 = T^r_r} = \frac{\epsilon(r)}{8\pi G}$ → let $T^0_0 = \frac{\epsilon}{8\pi G}$

$\boxed{T^\theta_\theta = T^\phi_\phi} = \frac{\mu(r)}{8\pi G}$ → let $T^\theta_\theta = \frac{\mu}{8\pi G}$

By ~~EFE~~ we get 2 Eqn

$\frac{1}{r^2}(1-f) - \frac{f'}{r} = \epsilon$ — ①

$\mu = \epsilon + \frac{r\epsilon'}{2}$ — ②

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Give any ϵ, find f, and find μ

③ Solving ①

$$f = 1 - \frac{a}{r} - \frac{1}{r}\int_a^r \epsilon r^2 dr \quad — ③$$

a is Int. const s.t. $a = r \Rightarrow f = 0$

Now let $\epsilon = 0$ Empty space solⁿ

If $\epsilon = 0$ then By ② $\mu = 0$

$\therefore T^\alpha_\beta = 0$ Empty space

In ③ if $\epsilon = 0$

$$f = 1 - \frac{a}{r}$$

$$= 1 - \frac{2GM}{r}$$

$$\frac{1-f}{r^2} - \frac{f'}{r} = \epsilon(r)$$

$$1 - f - f'r = \epsilon r^2$$

$$1 - \epsilon r^2 = f + f'r$$

$$1 - \epsilon r^2 = \frac{d(fr)}{dr}$$

$$\int_a^r (1 - \epsilon r^2)\, dr = fr$$

$$\frac{1}{r}\left[(r-a) - \int \epsilon r^2 dr\right] = f$$

$$\boxed{1 - \frac{a}{r} - \int_a^r \epsilon r^2 dr = f}$$

④ a const is choosen in such a way that
when $a = r$ $f = 0$
we could have choosen it in any othe way.

⑤ $$f = 1 - \frac{a}{r} + \frac{1}{r}\int^{r} \epsilon r^2\, dr$$

$\epsilon = 0$ (Empty space)

$$f = 1 - \frac{a}{r}$$

∴ consistent

⑥ Let ϵ = const.
then $\mu = \epsilon$
∴ T^i_i = const

But Earlier when $L = +k$
to a scalar field.
$$T^i_j = \mathcal{H} = \pi^i \partial_j \phi - \delta^i_j L$$

Constant er of $T^a_b = \delta^a_b k$

Equivalent.

⑦ Now let ϵ = const
Solution to $f = 1 - \frac{a}{r} \int_a^r \epsilon r^2 \partial r$

Assume No mass ∴ $T^a_{\text{mass}} = 0$ ∴ $\epsilon_{\text{mass}} = 0$
just T^a due to const is there.
$a = GM$. & ϵ^b is ther due to const.
and $M = 0 : a = 0$ let $a = 0$ $\Rightarrow$ $f = 1 - \frac{\epsilon r^2}{3} = 1 - H^2 r^2$
De Sitter Univers

de Sitter Universe

↳ when only const is added to eqn

⑧ $$ds^2 = (1 - H^2 r^2)\,dt^2 - \frac{dr^2}{1 - H^2 r^2} - r^2 d\Omega^2$$

⑨ Till Now we studied mass particle at origin

~~Now~~

lets have charge particle

$$E = \frac{q}{r^2}$$

$$\text{Energy Density} = \frac{E^2}{8\pi} \Rightarrow \frac{q}{r^4}$$

$$\epsilon = \frac{q}{r^4} \Rightarrow f = 1 - \frac{a}{r} + \frac{q}{r^2}$$

let $a = 2GM$ (from previous)

for charged particle ←

$$\boxed{f = 1 - \frac{2GM}{r} + \frac{q}{r^2}}$$

$$\hookrightarrow ds^2 = \left(1 - \frac{2GM}{r} + \frac{q}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{r} + \frac{q}{r^2}\right)} + \ldots$$

(RN) Ressnier Metric — for Electrostatic

We can also prove it on general grounds as we have proved for Sch. metric

⑩ EFE is Non Linear as $R_{ij} \approx \Gamma\Gamma$

$$\Gamma = \boxed{\partial_i g} : (\partial_i g)^2$$

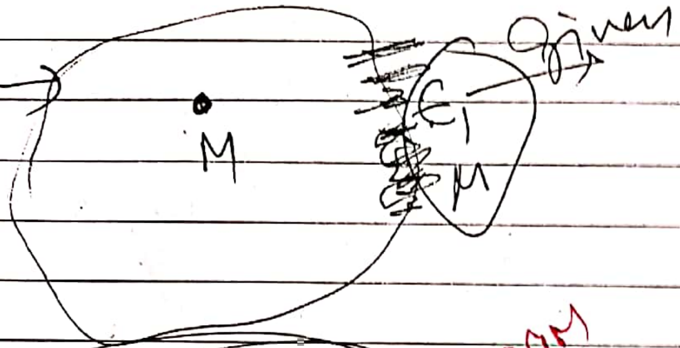
But $\frac{1}{r^2}(1-f) - \frac{f'}{r} = \epsilon \longrightarrow$ Linear inf

∴ Superposition ⇒ ∴ 2 sources ϵ_1 & ϵ_q

$\epsilon_1 + \epsilon_2$

Linear Eqn

$a_0(x)\, y + a_1(x)\, y' + a_2\, y'' - - - a_n y^n + b(x) = 0$



$$\frac{1-f_q}{r^2} - \frac{f_q'}{r} = \epsilon_q$$

$$\frac{1-f_d}{r^2} - \frac{f_d'}{r} = \epsilon_d$$

Non linear

??

$f_q = 1 - \frac{2qM}{r_2}$

$f_d = 1 - \mu^2 r^2$

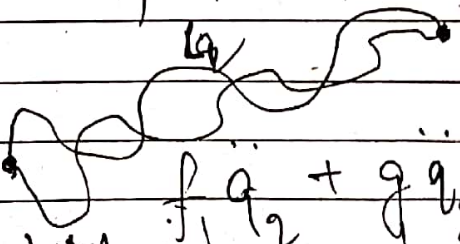
$F = g \cdot \mu^2 r^2 - \frac{2qm}{r}$

$$\frac{2-F}{r^2} - \frac{F'}{r} = E$$

Sol Q + Sol D = Sol

(11) $f_1 \ddot{q}_1 + g \mathring{q}_1 + h_1 = 0$ Lq

$f\, g\, h\, (q, \mathring{q}, t)$



$f_1 \ddot{q}_2 + g \dot{q}_2 + h_1 = 0$ Lm

$f \ddot{Q} + g \mathring{Q} + \mathring{h} = 0$

$\boxed{\text{Newtonian Eq a Linear Eq}}$

Interact $L \neq L_m + L_q$

Alex Flournoy L-15

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→ metric of Compact Star

① In this metric

$$ds^2 = f(r)\,dt^2 - f^{-1}dr^2 - r^2 d r^2$$

→ g_{ij} ind. of (t, ϕ)

∴ $K_t^\mu u_\mu = \tilde{E}$
$K_\phi^\mu u_\mu = \tilde{L}$ } const ⇒ $\tilde{L}$ = const ⇒ particle motion around Star in a plane

→ As its ind of (t, ϕ); $\frac{du_i}{d\tau} = \frac{\partial_i g_{\alpha\beta}}{2} u^\alpha u^\beta$

∴ u_0 is constant

for Both Photon & material particle

$u_0 = g_{00} u^0 = g_{00}\frac{dt}{ds} = \text{const} = k$

②
$$\boxed{\frac{dt}{ds} = \frac{k}{g_{00}}}$$

Now doing this for $K_t^\mu u_\mu = \tilde{E}$

→ Let Coordinate Basis be killing vectors in that frame also $K_t^\mu u_\mu$ = const would be valid

But $K_t^\mu \overset{*}{=} (1,0,0,0)$

(c) $u_0 = g_{00} u^0 = g_{00}\frac{dt}{d\lambda} = \tilde{E}$

ii) $u^i_{obs} p_i$ = Energy of photon

$(u^i_{obs} p_i)_1$ = Observer located at one pt.

$$(u_i p^i) = \tilde{E} = h\nu$$

ν → freq. of photon

~~$$\frac{(u^i p_i)_1}{(u^i p_i)_2} = \frac{\omega_1}{\omega_2} \quad \text{—} ①$$~~

→ Only for static metric

③ $u^i_{obs} \equiv \frac{1}{N(\vec{x})}(1,0,0,0)$ (Defn)

$\vec{x}$ → spatial

↳ for static metric

This guy is sitting quietly
∴ His spatial velocity shoul vanish

$$u^i u_i = 1 \Rightarrow g_{00}\frac{1}{N^2} = 1 \Rightarrow N^2 = g_{00}$$

N ← Lapse

in SR $\omega_1 = \omega_2$ as c is const for all frame in ①

here freq. changes

Same as u_0 is const

$$\frac{\frac{p_0}{N_1}}{\frac{p_0}{N_2}} = \frac{N_2}{N_1} = \frac{\omega_1}{\omega_2}$$

where $N_1 = N(\vec{x}_1)$ ← Spatial vector

$g_{\alpha\beta}(\overset{x}{t})$

$N_2 = N(\vec{x}_2)$ ← Spat

p_0 is conserved as $\frac{du_i}{d\lambda} = \frac{1}{2}\partial_i g_{\alpha\beta}u^\alpha u^\beta$

photon $u_0 = $ const.

~~p_0 is same (1,0,0,0)~~

∴ Diff is of N_1 & N_2 at Diff places of Observer

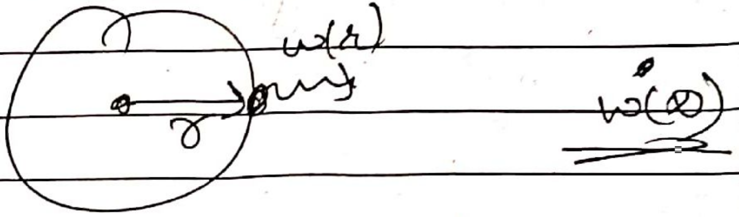
But u^2, u^3 are not const & $u^4_{photon} = $ const

(10 ④ $$\omega_\infty = \omega(r)\frac{N(r)}{N(\infty)} \qquad N(\infty) = 1$$

$$\omega_\infty = \omega(r)\left(1 - \frac{2GM}{r}\right)^{1/2}$$

∴ frequency of photon changes

if frq is $\omega(r)$ at



as r reaches $r_s = 2GM$

$\omega_\infty \downarrow\downarrow$ ∴ Red shift.

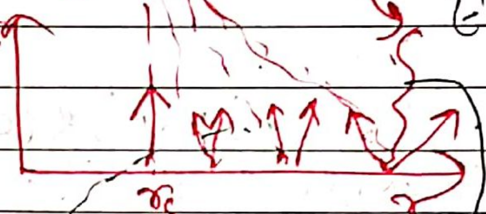
Examples

in Schwarchild geometry i.e. $f(r) = 1-\frac{2GM}{r}$

Photon

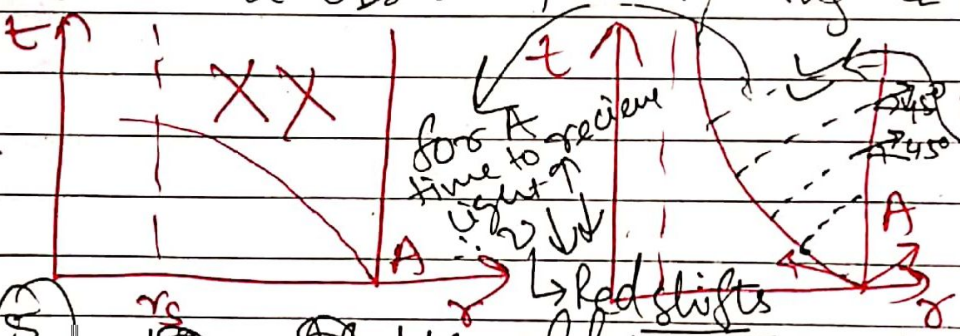
$$\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}$$

in Sch. coord



Every photon going asympt

Now let 2 Observer, one moving & other at Rest



for A to recieve time to light r ↓↓

↳ Red shifts

Because particle is timelike always (in b/w light cones)

$$d\tau_A^2 = \left(1-\frac{2GM}{r_1}\right)dt^2$$

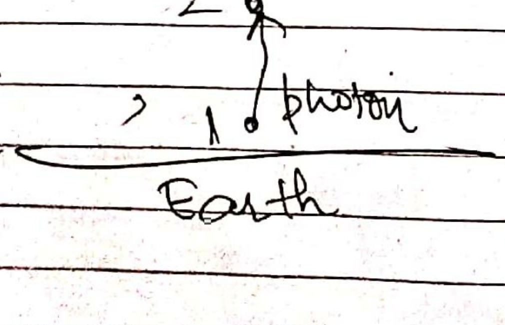
$$d\tau_B^2 = \left(1-\frac{2GM}{r_2}\right)dt^2$$

$$\boxed{\frac{d\tau_A}{d\tau_B} = 1+GM\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}$$

(S) r_s ω_∞ Usually $r > r_s = 2GM$

∴ $\left(1-\frac{2GM}{r}\right) < 1$

∴ $\omega_\infty \downarrow\downarrow \Rightarrow \omega_\infty < \omega_r$



then freq of photon at 2 is less than 1

∴ Energy of photon dec.

in lecture ⑦
Bring/Derive that Result from this

⑥ Even with Cosmological case, you can have complete static metric.

⑦