L-6

$\partial_i \pi^i = \frac{\partial \mathcal{L}}{\partial \phi} \longrightarrow \Box \phi = -\frac{\partial V}{\partial \phi}$

(1) $\partial_a F^{ab} = 4\pi J^b = \partial_a (\partial^a A^b - \partial^b A^a)$

$= \partial_a \partial^a A^b - \partial^b \partial_a A^a$ $\qquad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

(2) $\partial_a \partial^a = \eta^{ab} \partial_a \partial_b = \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$

wave operator

$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

$\therefore \; \partial_a F^{ab} = \Box A^b - \partial^b(\partial_a A^a) = 4\pi J^b$

(3) if $\partial^b (\partial_a A^a) = 0$

then

$\Box A^b = 4\pi J^b$ (wave eqn in ED)

(4) $\partial_a F^{ab} = 4\pi J^b = \Box A^b - \partial^b(\partial_a A^a)$

A^b cannot be found due to Gauge transfn

& $F^{a'b'} = F^{ab}$

& we know only about F^{ab} from $\vec{E}$ & $\vec{B}$ $\therefore$

A^b cannot be uniquely found.

(5) Let $(\partial_a A^a) = \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$ $\qquad$ choose the Gauge like this

Divergence of A

Lorentz Gauge

(6) ~~$A^a_a = 0$~~ $\qquad F_a{}^a = 0$

$F_a{}^a = \partial_a A^a - \partial_a A^a = 0$

But we are putting $\partial_a A^a = 0$

(7) $\therefore \Box A^k = 4\pi J^k$ — As this is the linear eqn. → $\Box\phi = -\frac{\partial V}{\partial \phi}$ — As was the Assumption of Superposition told us.

(8) Fourier Transf.

$$A^k(x^j) = \int \frac{d^4 p}{(2\pi)^4} A^k(p) e^{ipx}$$

$ip_j x^j$ → $p_j x^j$

$\Box A^k = 4\pi J^k$ Becomes as $\partial_a \partial^a e^{ip_j x^j} \Rightarrow -p^j p_j$

(9) $-p^j p_j A^k(p) = 4\pi J^k(p)$

(10) Whenever you have linear eqn do fourier transform & in fourier space derivative operators just become algebraic operator.

$$\therefore \quad A^k(p) = -\frac{4\pi}{p^2} J^k(p)$$

(11) $\therefore$ I have J^k — (1) do fourier Transform & get $J^k(p)$

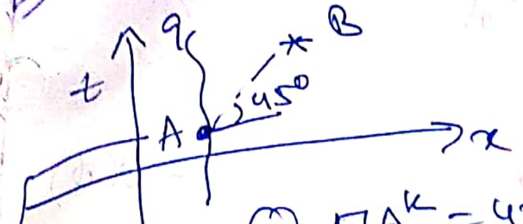
(2) $\frac{-4\pi J^k(p)}{p^2}$

(3) Inv. fourier Transform to get A^k.

(12) $\Box F_{ik} = 4\pi (\partial_i J_k - \partial_k J_i)$ from (7)

$\therefore F_{ik}$ can be express as first Derivatives of 4-current.

(3) wave ... propagating at c speed.



let charge particle be moving in spacetime

then by (7) $\Box A^k = 4\pi J^k$

it tells the influence of the charge J^k on field has to propagate at speed of light.

↳ what charge is doing at A pt will lead to field at B.

(15) What prop of traj. can effect B pt?

(2) by

$\Box F_{ik} = 4\pi(\partial_i J_k - \partial_k J_i)$

Source involves first Derivative of current

& as current is linear in $\vec{v}$

∴ Max double derivative of $\vec{x}$ w.r.t. spatial & time coordinate can happen.

~~$\vec{E}(x^i, \dot{x}^i, \dddot{x}^i)$~~

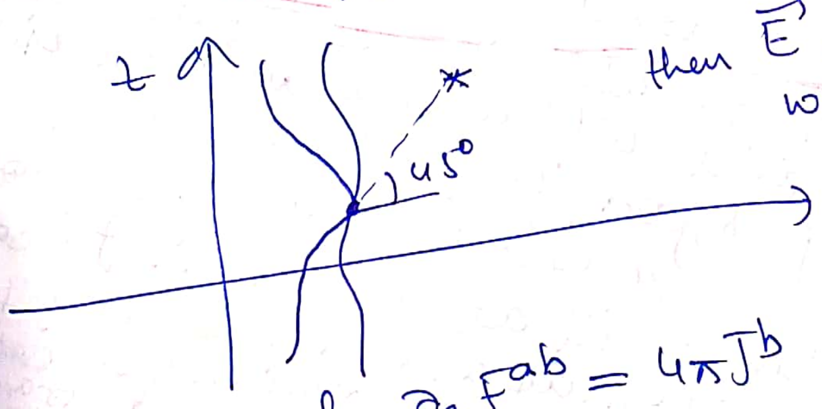
∴ $\vec{E}$ at B pt depends on $\vec{E}(x^i, \partial_i x, \partial_i \partial_j x)$

Not on derivative of acc. (x^i of charge particle)

(3) ∴ if we have 2 traj which have same pt., $\vec{v}$, $\vec{a}$ at same pt.

i.e. curvature, tangent

then $\vec{E}, \vec{B}$ along null line would be same for both.



(7) we obtained $\partial_a F^{ab} = 4\pi J^b$ by Gauge invariance & then imposing gauge getting $\Box A^k = 4\pi J^k$

(18) $L_{field} = \frac{1}{2} \partial_i A_j \partial^i A^j + 4\pi J_k A^k$ (without invoking F_{ab}) 74

Similar to scalar field → $L_{scalar\ field}$ → But $L_{scalar\ field} = \frac{1}{2} \partial_a \phi \partial^a \phi - U(\phi)$

(19) L-5 (40)

$$\frac{\partial L}{\partial \phi} = \partial_i \left(\frac{\partial L}{\partial(\partial_i \phi)} \right)$$

$$= \partial_i \partial^i \phi = \Box \phi$$

$$-\frac{\partial U(\phi)}{\partial \phi} \quad \frac{\partial L}{\partial \phi}$$

if $U(\phi) = 0$ then $\Box \phi = 0$

(20) EOM for Vector field

$$L(\phi, \partial_a \phi) = \frac{\partial_a \phi \partial^a \phi}{2}$$

$$= \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial t}\right)^2 - (\nabla \phi)^2\right)$$

$$\frac{\partial L}{\partial A^k} = \partial_i \frac{\partial L}{\partial(\partial_i A^k)}$$

$$\Box A^k = 4\pi J^k$$

To get this result originally what we did $F_{ab}F^{ab}$ & then write down field eq^n → Gauge inv. → $\Box A^k = 4\pi J^k$
But Now we got to this by L_{field} in (18) → Though this would give field eq^n But

(21) $L_{field} = \frac{1}{2}(\partial_i A_j \partial^i A^j) = \frac{1}{2}\left((\partial_t \phi)^2 - (\partial_t A)^2\right) - \ldots$

↑ wrong sign

Compare with Scalar field

By using this Energy would come negative Energy.

(22) whenever we have Gauge fields like vector field here there are dof of the field which apparantely carries -ve energy ∴ theory is good if we have way of eliminating those -ve energy carrying modes ∴ from 4 Dof I should be able to kill atleast one ∴ I need Symmetry Transf^n involving atleast one fn & that is what Gauge Transf^n gives. Now in our case as Gauge Transf existence can tc of -ve energy mod

23. Maxwell Eqn. was not invariant under Galilean Transfn.
So Lorentz found eqn under which it was invariant.

24. If we have mechanical system coupled to EM field. Then we run into trouble as
mechanical system are inv. under Gal. Transf.
Maxwell Eqn under Lorentz Transfn.

25. Eistein said everything follows Lorentz Transfn.

26. Newton's Gravity

$L = \frac{mv^2}{2} - U(\vec{r}) = \frac{mv^2}{2} - m\phi(\vec{r})$

$U(\vec{r}) = -\frac{GmM}{r}$ $\quad \phi(\vec{r}) = -\frac{GM}{r}$

Particle in Scalar field

$\rightarrow m\frac{d^2x}{dt^2} = -m\vec{\nabla}\phi$ $\quad$ NR $\frac{d\vec{p}}{dt} = -\vec{\nabla}\phi$

ϕ: Gravitational Potential

ϕ can be found by

Dynamics of scalar field $\Box\phi = -\frac{\partial V}{\partial \phi}$

Dy. of vec field $\Box A^k = 4\pi J^k$ assuming gauge $\partial_a A^a = 0$

Poisson Eqn: $\rightarrow \nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x})$ $\quad$ ρ: mass density

$\therefore$ this tells that if ρ is changed then instantaneously ϕ changes.

This Eqn is supposed to be true if $\phi(t,\vec{x})$ & $\rho(t,\vec{x})$.

→ For static fields $\Box A^k = 4\pi J^k \Rightarrow \nabla^2 A = 4\pi J^k$
Time Derivative vanishes
→ Poissons Eqn
→ Like charge attract/Repel?
→ $\overset{+}{o}$ $\quad$ o^+ $\quad$ E

$\overset{+}{o}$ $\quad$ $\overset{+}{o}$ $\quad$ E′

if $E' < E$
$\therefore$ Repel.

→ In Scalar field like charge attract.
→ Tensor Rank 0 like charge attract
Tensor Rank 1 like charge repel
Tensor Rank 2 like charge attr
→ Raychaudhuri Eqn ?

(27) So $\nabla^2 \phi = 4\pi G \rho$ — then what about charge density L-5 (21)

$\nabla^2 \longrightarrow \Box$

& as ρ is mass density & ∴ it is not Lorentz inva.

∴ RHS has to be changed to rel. invariant object which reduced to this ρ in NR.

(28) We can do this But that doesn't agree with Exp. bending of light.

(29) Poisson Eqn for gravity $\nabla^2 \phi = 4\pi \rho G$

$\nabla^2 \to \Box$.

Now ρ has to be replaced. The natural replacement is $T^a{}_b$ = Energy Momentum Tensor

∴ ϕ also has to be replaced by 2nd Rank Tensor. h^{μ}_{ν}

(30) ∴ $\Box \left(- \frac{\bar{h}_{ij}}{4}\right) = 4\pi G T_{ij}$ — But this theory doesn't agree with bending of light

This is fully Covariant form of Poisson's Eqn for Gravity.

(31) $ds^2 = \eta_{ab} dx^a dx^b$

in presence of gravity

$ds^2 = g_{ab}(t, \vec{x}) dx^a dx^b$

g_{ab} Symmetric

g_{ab} will tell what are the symmetries of the space eg. $ds_g^2 = ds^2(1+\phi_g)$

as one set of 6 determines other 6

g_{ab} are 10 fn — 4 Diag, 6 Non Diag.

(32) In Newtonian theory only one fn ϕ is given $\nabla^2 \phi = 4\pi G \rho$

But in GR 10 fn are given to describe Gravity

① $m\frac{d^2\vec{x}}{dt^2} = -m\vec{\nabla}\phi$ ⟶ $\mathcal{L} = \frac{m\dot{x}^2}{2} - m\phi(x)$

$\nabla^2\phi = 4\pi G\rho$ (Poisson's Eqn)

→ if position of 1 particle is changed then all others particles change instant.

∴ STR violated

② Newton's 2nd law is the Basic law. everything all other laws could be derived from it.

$m\frac{d^2x}{dt^2} = m\ddot{x} = -\frac{\partial U(x)}{\partial x} = F$

$\frac{d\vec{p}}{dt} = \vec{F}$

① let ~~p = const~~ ⇒ F = 0 ⇒ p = const.

0 force leads to const. speed.

③ let p = const.

$F_{12} + F_{21} = 0 \Rightarrow F_{12} = -F_{21}$

When transl. Invariance or is there

$\frac{\partial L}{\partial r} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}} = \text{const} = P$

↓

$F_{12} + F_{21} = 0$

(33) This modification has very drastic implication for geometry.

This makes flat space ⟶ Curved space.

(34) Just like EM, we should work that out
But as in (28) this doesn't match Exp.
Therefore we have to change geometry. But why?

$$ds^2 = \eta_{ab}\, dx^a dx^b \longrightarrow g_{ab}(\vec{x}, t)\, dx^a dx^b$$

(35) $ds^2 = g_{00}(\vec{x}, t)\, dt^2 + \cdots$

Let $dx, dy, dz = 0$ then $ds^2 = g_{00}(\vec{x}, t)\, dt^2$.
Let there be 2 clocks at different place in space
then Rate of both of them would be different.

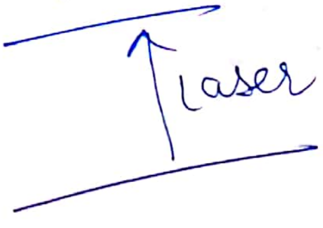
(36) In SR if 2 clocks are moving w.r.t each other they move at different rates.
But if both are stationary in a frame in SR the clock rate is same.
But in GR if 2 clocks are in same frame stationary but at different places their clock rate differs.
i.e. Gravity effects clock rate.

(37) If I prove that gravity effects clock rate then ↳ then space is not hom. How?

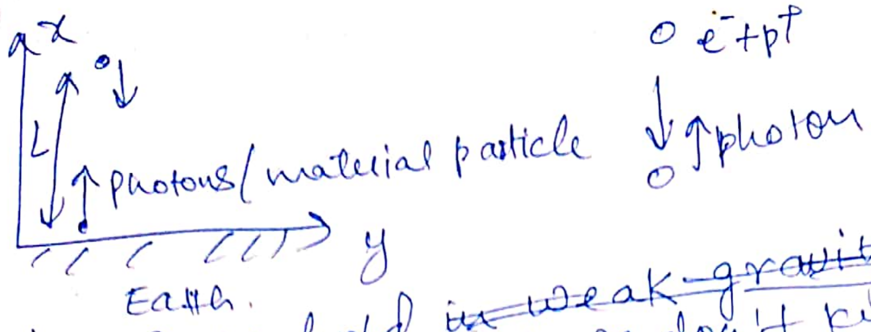
$\Rightarrow ds^2 = g_{00}(\vec{x}, t)\, dt^2$

? And now if I want to maintain SR invariant then modify $ds^2 = g_{ab}(\vec{x}, t)\, dx^a dx^b$.

(38) To prove: If I have radiation of ν freq. & it is moving in gravity eg. then that ν freq. has to change.



(39)



~~in weak gravity~~.

Assuming SR is hold — But we don't know if 2nd post is Right or not

$2h\nu \longrightarrow 2mc^2$

photon $\qquad e^- - p^+$ pair

Where we used it?

Assuming particle gains energy mgL when moving down

$E_{down} = E_{up} + mgL$ ~~Why mgL~~

$\qquad\qquad\quad$ "

Assuming photon freq. mc^2 remains same

$h\nu_{down} = E_{down}$

∴ Each time gain energy

But this can't happen due to $\phi(x)$

Static gravity — Energy Conservation as $L(x,t)$?

time invariant

converting at top Photon ↓ $p^+ + e^-$

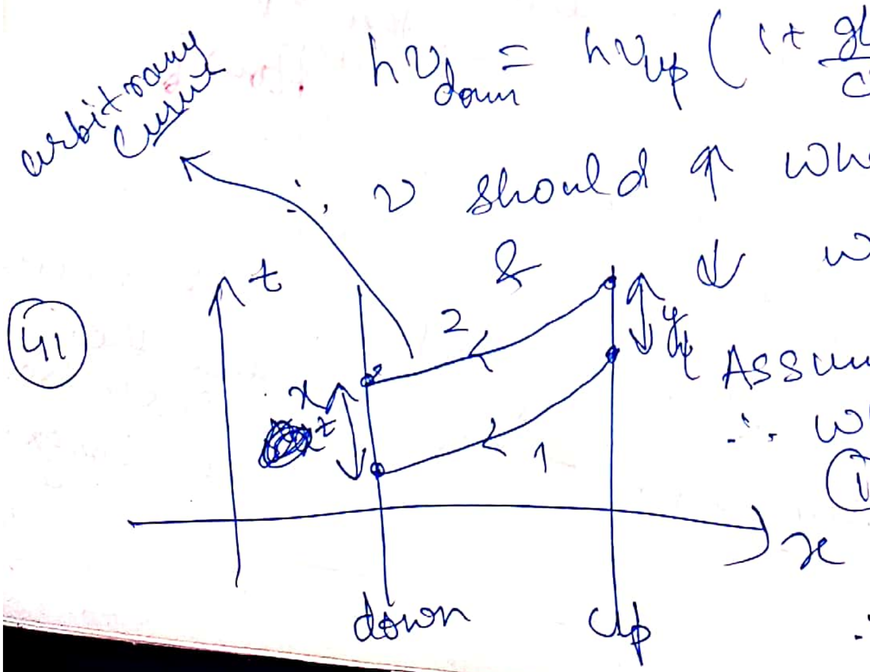
(40) ∴ $h\nu_{down} = h\nu_{up} + \frac{h\nu_{up}}{c^2} gL$

$h\nu_{down} = h\nu_{up}\left(1 + \frac{gL}{c^2}\right)$

∴ ν should ↑ when coming down

& ↓ when going up

(41)



Assuming static gravity

∴ whatever happens to ① same happens to ②

∴ $x_t = y_t$

(42) freq ... by down observer

$\nu_{down} = \frac{N}{\Delta t_{down}}$ = No. of times photon came to him

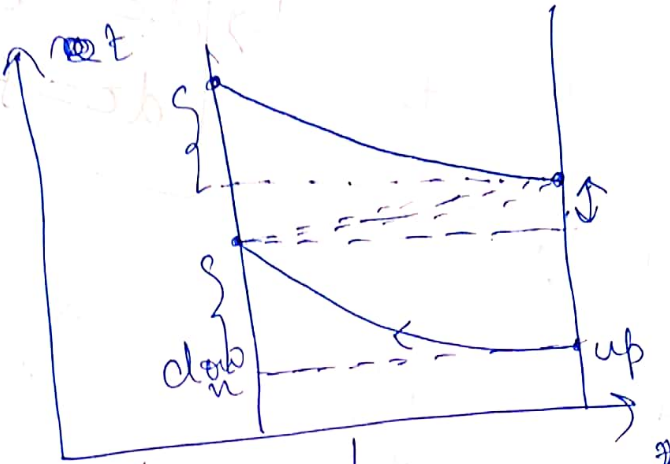
Δt_{down} = time obs. by down obs.

$\nu_{up} = \frac{N}{\Delta t_{up}}$

By (40)

$\therefore \frac{N}{\Delta t_{down}} = \frac{N}{\Delta t_{up}}\left(1 + \frac{gL}{c^2}\right) \Rightarrow \Delta t_{down} = \Delta t_{up}\left(1 - \frac{gL}{c^2}\right)$

(43)



let N = 1

$\frac{1}{\Delta t_{down}} = \frac{1}{\Delta t_{up}}\left(1 + \frac{gL}{c^2}\right)$

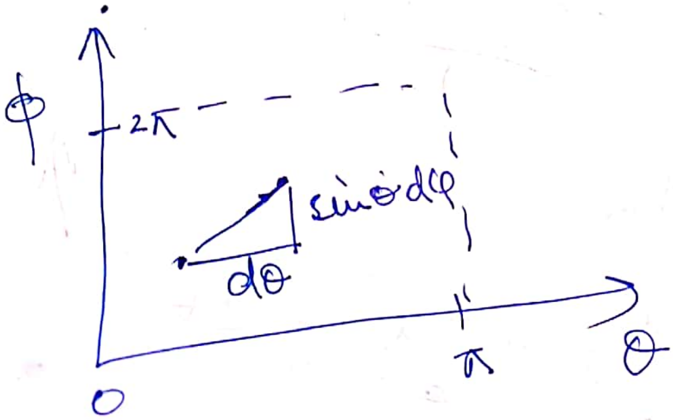
But $\Delta t_{down} = \Delta t_{down}$ from geometry

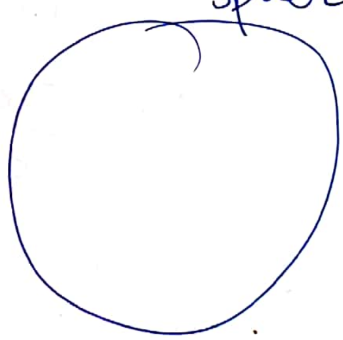
$\therefore$ Contradiction.

$\downarrow$

This interpretation is wrong.

(44)





length interval

$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$

let radius be const.

$d\vec{r} = r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$

$g(d\vec{r}, d\vec{r}) = r^2(d\theta^2 + \sin^2\theta\, d^2\varphi)$

In Cartesian

$$dr^2 = dx^2 + dy^2 + dz^2$$

here $dr^2 = r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$

$\sin^2\theta \to f(\theta)$

$\therefore$ Physical length is $\sin\theta \, d\varphi = f(\theta) \, d\varphi$

Similarly

(45) Physical time interval $\neq$ Geometrical time int. which is Δt_{down}

$\hookrightarrow$ $d\tau = e^{\int 2gx'/c^2} dt'$

$d\tau = f(x) \, dt'$

Infact

Physical time int. $= f \, \Delta t$

(46) $\Delta t_d = \Delta t_u \left(1 - \frac{gL}{c^2}\right)$

$\uparrow x$

As $-mg = -m\vec{\nabla}\phi = -\frac{\partial \phi}{\partial \vec{x}}$

& Assuming $\phi_u = mx + \phi_d$ "linear"

$\therefore \frac{\partial \phi}{\partial x} = K = -\frac{(\phi_d - \phi_u)}{L}$

$\therefore mgL = \phi_u - \phi_d$

(47) $\Delta t_d = \Delta t_u \left(1 - \frac{\phi_u}{c^2} + \frac{\phi_d}{c^2}\right) = \Delta t_u \left(1 - \frac{\phi_u}{c^2}\right)\left(1 + \frac{\phi_d}{c^2}\right)$

we can keep ϕ_{up} anywhere

$(\Delta t_d)^2_x = (\Delta t)^2_{fid} \left(1 + \frac{2\phi_x}{c^2}\right)$

Assuming $\phi_{fid} \to 0$ Reference point ∴ fid. pt. is where $g \to 0$ 81.

(48) ∴ $$(\Delta t)_x^2 = (\Delta t)_{fid}^2 \left(1 + \frac{2\phi_x}{c^2}\right)$$

if change in potential is Not linear.

On Earth
$$(\Delta t)_x^2 = (\Delta t)_{fid}^2 \left(1 - \frac{2GM}{Rc^2}\right)$$ → gravity slows down time.

when $R \to \infty$
$$(\Delta t)_R \Big|_{R\to\infty} = (\Delta t)_{fid}$$
clocks located at ∞ which is not in any Gravitation field

(49) ∴ Physical time int $= \left(1 + \frac{2\phi_x}{c^2}\right)(\Delta t)_{fid}^2$ as in (44).

$c = 1$ ∴ $g_{00}(t, \vec{x}) = (1 + 2\phi)$

why others have to change ?

(50) ∴ if Energy Conservation has to hold then Grav. field has to effect flow of time. & as lengths are measured as by ~~speed~~ Light & clocks ∴ It will effect ~~meas~~ length interval also

(51) Principle of Equivalence

$m \frac{d^2x}{dt^2} = -m \vec{\nabla}\phi$

m is the inertia of the body; m (right side) — how the particle is coupling to Aij Gravitational field "Gravitation charge"

$m \frac{d^2x}{dt^2} = -q \vec{\nabla}\phi_{Elect}$

m is the inertia of body; q — how the particle is coupling to Aij given by q, "Electric charge"

q can be +, −, neutr

Similarly particle may ∃ which do not couple to Grav. field

∴ There can be particle which respond [to gravity] but no EM.

Similarly There can be particle which respond to EM but no Gravity.

(52) But it doesn't happen. (that any mass particle will interact with grav. force)

Bcz. Gravitational charge or Gravitational mass (m_g) = inertial mass (m_i)

$$m_g = m_i$$

(53) ∴ $m \frac{d^2x}{dt^2} = - m \vec{\nabla} \phi \Rightarrow \frac{d^2x}{dt^2} = - \vec{\nabla} \phi$

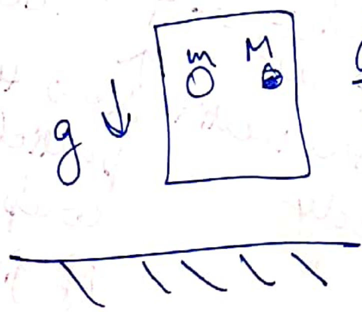
∴ If we have heavy ball & light ball they both will hit the grnd at same time.

⇒ The traj. of Particle with same initial condition are independ. of all the prop. of the particle.

⇒ If $\phi_{grav.}$ is given how any particle would move can be told.

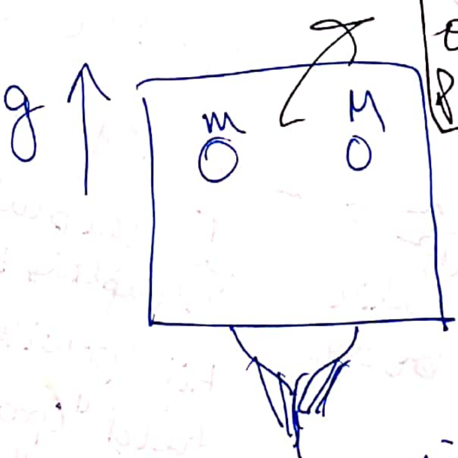
∴ Traj. of particles are ind. of their prop.
"Principle of Equivalence"

(54)



$\frac{d^2x}{dt^2} = g$

$x = ut + \frac{gt^2}{2}$



What is the origin of pseudo force?

Simultaneously they would hit if initial cond. is same

No force acting on m & M & ∴ floor hits simult.

~~In~~ 2nd case we can think g acting downward & both hit simult.
So locally in small region both cases are indistinguishable. Why locally? See (61)

In a weak gravitation field ⇒ L.I action ∴ $ds^2 = c^2dt^2 - d\vec{x}^2$

(55) (1) $A = -mc^2\int ds - m\int \phi\, ds = -mc^2\int\left(1+\frac{\phi}{c^2}\right)ds$ → Assuming ϕ is L.I.

Similar to scalar field and in NR it would reduce to

$$\frac{d\vec{p}}{dt} = -m\vec{\nabla}\phi \quad \& \quad L = -mc^2 + \frac{mv^2}{2} - m\phi$$

⇓

(56)

$$\frac{d\vec{u}}{dt} = -\vec{\nabla}\phi$$ EOM is ind. of prop. of particle.

(57) $A = -mc^2\int\left(1+\frac{\phi}{c^2}\right)ds$

When $v \ll c$
$ds^2 = c^2dt^2 - |d\vec{x}|^2$

$ds\left(1+\frac{\phi}{c^2}\right)$

Square ~~& multiply c^2~~

$$ds^2\left(1+\frac{\phi}{c^2}\right)^2 \Rightarrow \left(c^2dt^2 - |d\vec{x}|^2\right)\left(1+\frac{\phi}{c^2}\right)^2$$

$$\Rightarrow \left(1+\frac{\phi}{c^2}\right)^2 c^2dt^2 - |d\vec{x}|^2\left(1+\frac{\phi}{c^2}\right)^2$$

$v \ll c$

$$\Rightarrow (c^2+2\phi)\,dt^2 - |d\vec{x}|^2$$

$g_{ab}(\vec{x},t)dx^a dx^b$ — ds_g^2 ~~$ds^2\left(1+\frac{\phi}{c^2}\right)^2$~~ $\Rightarrow \left(1+\frac{2\phi}{c^2}\right)c^2dt^2 - |d\vec{x}|^2$

$\Rightarrow g_{00}\, c^2dt^2 - |d\vec{x}|^2$

∴ $A = -mc^2\int ds_g$ — (2)

where $g_{00} = 1 + \frac{2\phi}{c^2}$

Metric is changed

∴ 2 ways of seeing (1) On weak G.R
(2) Metric is changed → geometry change, No gravity field

See (49)

(58) Doing the same thing for scalar field

$$A = -\lambda \int \phi \, ds$$

then

(5:
$$A = -mc^2 \int ds - \lambda \int \phi \, ds$$

$$= -mc^2 \int \left(1 + \frac{\lambda \phi}{mc^2}\right) ds$$

Geometry is not changed

$$\therefore g_{00} = 1 + \frac{2\lambda\phi}{mc^2}$$

geometry depends on λ & m

(5: $\therefore$ Different particles see different geometry?

$\Rightarrow$ Artificial way of talking about things

(59) $\therefore$ in (57) Crucial fact used is the that of EQP. m is same.

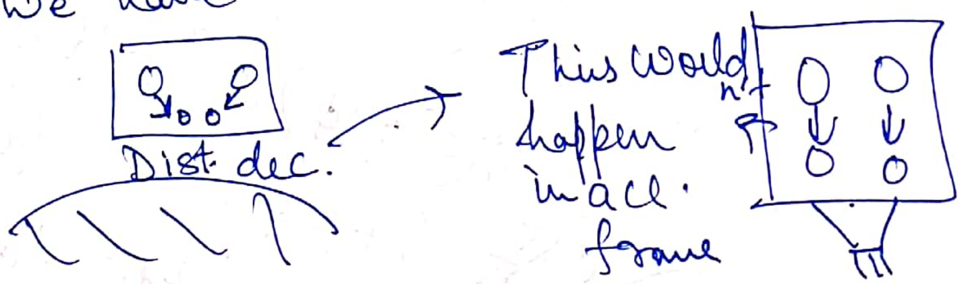
$\Rightarrow$ All m sees same geometry.

(60) By (54)

in acc. frame also $g_{00} = \left(1 + \frac{2\phi}{c^2}\right)$

(61) in (54) we said locally acc = gravity

if we have



$\therefore$ Gravity is not completely Equivalent to acc. frame

This is the next higher order effect which leads to curvature.

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(1) Time Interval is diff. for diff. sys in gravity.

Proof: → Energy conservation
→ ν change of photon
→ Time change

(2) $d\tau^2 = f(x)\, dt^2$ & $g_{00} = \left(1 + \frac{2\phi}{c^2}\right)$ in weak grav.

Proof: → at ∞ $\phi(x) = 0$ & gravity is static

(3) $d\tau^2 = g_{\alpha\beta}(x,t)\, dx^\alpha dx^\beta$

Proof: → if gravity is Not static
→ Scalar for Action

(4) Equiv. Principle ⇒ acc. frame $d\tau^2 = \left(1 + \frac{2\phi}{c^2}\right) dt^2$

(5) weak Grav = Scalar field in SR action

Proof: → Scalar field in SR
→ Eq. principle
→ $\left(1 + \frac{2\phi}{c^2}\right)$

Scalar field for SR ⇒ geometry change

(7) why no geometry change for EM field.

Doubts

(?) & (5) are in circular arguments.

$L = \frac{mv^2}{2} \rightarrow L = mV$
Only Lag has change $L(q)$ ∴ E conser

Energy cons. in SR of free particle
→ Energy cons. in SR of particle in Scal & EM field.

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① In NR

Free particle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{d\vec{p}}{dt} = 0$$

$$\frac{d\vec{v}}{dt} = 0$$

$$v = k$$

$$\boxed{x = kt}$$

In SR

free particle

$$\frac{\partial u^i}{\partial \tau} = 0 \Rightarrow \frac{d(\gamma \vec{v})}{dt} = 0$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = 0 \Rightarrow \frac{d(\gamma \vec{v})}{dt} = 0$$

$$\gamma v = k$$

$$\frac{\left(\frac{dx}{dt}\right)}{\sqrt{1 + \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}} = k$$

$$\Rightarrow \dot{x}^2 = k^2 + \frac{k^2}{c^2}\dot{x}^2$$

$$\dot{x}^2\left(1 - \frac{k^2}{c^2}\right) = k^2$$

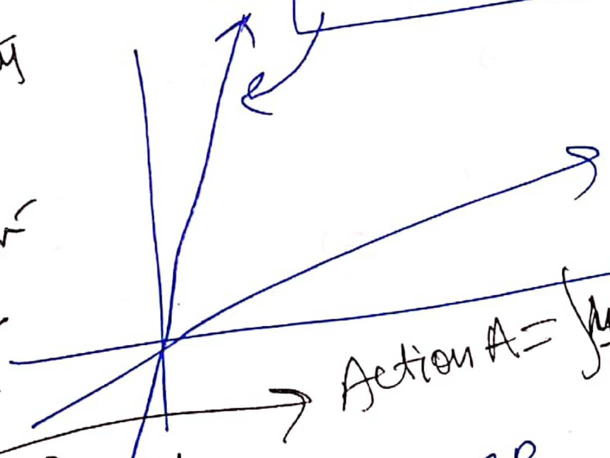
$$\dot{x} = \frac{k}{\sqrt{1 - \frac{k^2}{c^2}}} \Rightarrow x = \frac{kt}{\sqrt{1 - \frac{k^2}{c^2}}}$$

$$\therefore \boxed{x = \gamma v t}$$

① Action for Acc. particle in SR?

② EOM for acc. motion by $a_i a^i = -a^2$

Obtaining NR velocity expression $v = at$ & acc. expr $x = \frac{at^2}{2}$



Action $A = \int \frac{mv^2}{2} dt + \int \lambda \phi \, dt$

$$L = \frac{mv^2}{2} + \lambda \phi$$

In NR

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x} \Rightarrow \frac{d\vec{p}}{dt} = m\frac{\partial \phi}{\partial x}$$

$$\therefore \frac{d\vec{v}}{dt} = g \Rightarrow \boxed{x = \frac{gt^2}{2}}$$

In SR

for free Particle action $A = -\int m \, ds$

some force $A = -m\int ds - \lambda \int \phi \, ds$

Const. acc. → How?

EOM

$$\boxed{\frac{d}{dt}(\gamma v) = g}$$

9

$$gt = \frac{dx/dt}{\sqrt{1-\frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}}$$

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}$$

let $u = \frac{g^2 t^2}{c^2} + 1$

$du = \frac{2g^2}{c^2} t \, dt$

$\Rightarrow dx = \frac{gt \frac{c^2}{2g^2 t} du}{\sqrt{u}}$

$dx = \frac{c^2}{2g} \frac{1}{\sqrt{u}} du$

~~$x = \frac{c^2}{g} \sin$~~

$x = \frac{c^2}{g} \sqrt{\frac{g^2t^2}{c^2}+1}$

$\therefore \frac{x^2 g^2}{c^4} = \frac{g^2 t^2}{c^2} + 1$

$$x^2 - c^2 t^2 = \left(\frac{c^2}{g}\right)^2$$

$c^2t^2 - x^2 = -\left(\frac{c^2}{g}\right)^2 \leftarrow$

$x = \frac{c^2}{g} \cosh f(\tau)$

$ct = \frac{c^2}{g} \sinh f(\tau)$

$y = c \cosh f(\tau)$

$x = c \sinh f(\tau)$ $\Leftarrow$ $y^2 - x^2 = c^2$

Hyperbola equ-

$\therefore x = \frac{c^2}{g} \cosh f(\tau)$

$ct = \frac{c^2}{g} \sinh f(\tau)$

for pt. charge if $f=0$ at Boundary + $\partial_k J^k = 0$ to make it 4-I ⇒ Charge conse
for charge density if $f=0$ at surface
A is LI → L is LI. → A is not 4-I. → L is not 4-I → EOM is 4I.

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particle Coupled to Vector field

1) $A = -m\int ds - q\int A_j dx^j \quad A^j = (\phi, \vec{A})$

2) $L = -\frac{m}{\gamma} - q\phi + q(\vec{A}\cdot\vec{v}) \quad L_{NR}$

$p = \frac{\partial L}{\partial v} = m\gamma\vec{v} + q\vec{A} \quad H = pq - L = m\gamma + q\phi + q(\vec{A}\cdot\vec{v})$

③ $\delta A = -\int d((mU_i + qA_i)\delta x^i) + \int (m\frac{dU_i}{ds} - qF_{ij}U^j)\delta x^i ds$

④ $\frac{dp^i}{ds} = qF^i_j U^j \quad \vec{E} = (F_{01}, F_{02}, F_{03})$ we get $\vec{\nabla}\times E = -\frac{\partial \vec{B}}{\partial t}$

$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad \vec{\nabla}\cdot\vec{B} = 0$

$(F^*)^{ab} = \epsilon^{abcd}F_{cd} \quad \partial_a (F^*)^{ab} = 0$

⑤ Rel. Lorentz force $\quad v \ll c$

$\frac{d\vec{p}}{dt} = \frac{d(m\gamma\vec{v})}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \longrightarrow \frac{dm\vec{v}}{dt} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$

↳ $\frac{dp^0}{dt}$ Redundant.

⑥ $\frac{\delta A_c}{\delta x^i} = -P_i = -(mU_i + qA_i) = (-E, \vec{p})$

⑦ Def. $P_i = (mU_i + qA_i) = $ 4 mom. $= (E, -\vec{p})$

⑧ $\frac{\delta A_c}{\delta x^0} + E = 0 \quad \frac{\delta A_c}{\delta \vec{x}} = \vec{p}$

⑨ $(\partial_i A_c + qA_i)(\partial_j A_c + qA_j)\eta^{ij} = m^2$

10) 4.T. $A'_j = A_j + \partial_j f \Rightarrow F'_{ij} = F_{ij}$

11) $E'_{\parallel} = E_{\parallel} \quad E'_{\perp} = \gamma(E_{\perp} + (\vec{v}\times\vec{B})_{\perp}) \quad B'_{\perp} = \gamma(B_{\perp} - (\vec{v}\times\vec{B})_{\perp})$
$B'_{\parallel} = B_{\parallel}$

12) $A = -m\int ds - \int A_k J^k d^4x \quad J^k = (\rho, \rho\vec{v}) = (\rho, \vec{j}) \quad \rho \equiv \sum_i q_i \delta_D(x^i - z^i)$

A Demand 4.I ⇒ $L(F_{ab})$

Dynamics of Vector field

① $A = -m\int ds - \int A_k J^k d^4x - \int L_f d^4x \quad L_f(A_j, \partial_k A^j)$

Due to A being 4-I. $L_f(F_{ab})$ Due to Lorentz inv. $L_f = F^{ab}F_{ab}$, $\epsilon^{abcd}F_{ab}F_{cd}$

$A_f = \frac{-1}{16\pi}\int d^4x\, F^{ab}F_{ab}$

② $\delta A = -\frac{1}{4\pi}\int \partial_i(F^{ik}\delta A_k)d^4x + \int(\frac{1}{4\pi}\partial_i F^{ik} - J^k)\delta A_k d^4x$

$\therefore \partial_i F^{ik} = 4\pi J^k$ — ϕ vague dof.

④) $\partial_i F^{ik} = 4\pi J^k \equiv \vec{\nabla}\cdot\vec{E} = 4\pi\rho, \quad \vec{\nabla}\times\vec{B} = 4\pi\vec{j} + \frac{\partial\vec{E}}{\partial t}$; $\partial_i F^{*ik} = 0$

③ $\delta A_c = \frac{-1}{4\pi}\int d^3x\, F^{0k}\delta A_k = \delta A \propto \int d^3x(\vec{E}\cdot\delta\vec{A})$ → wave eqn

④) $\partial_i F^{ij} = \Box A^j - \partial^j(\partial_i A^i) = 4\pi J^j$ if $(\partial_i A^i) = 0 \Rightarrow \Box A^j = 4\pi J^j$

$\frac{\partial}{\partial t}\phi - \vec{\nabla}\cdot\vec{A} = 0$

constraint on $J^k \quad \partial_k J^k = 0$
cons.

It tells influence of charge J^k on field has to propagate at speed of light.

$\Box F_{ik} = 4\pi(\partial_i J_k - \partial_k J_i)$ ⑥ $L_f = \frac{\partial_i A_j \partial^i A^j}{2} + 4\pi J_k A^k \Rightarrow$ Neg. Energy

$\vec{E}\,(x^i, \partial_i x, \partial_i\partial_j x)$

(38) Poisson's Eqⁿ $\nabla^2\phi = 4\pi G\rho \equiv$ Grav.
$\nabla^2 A = 4\pi J^a \equiv$ Static EM field
$\Box \longrightarrow \nabla^2$

(38) Grav. field should have (4) Newton's Approx.
(1) Lorentz Invariant (2) Grav. Force must be attract (3) Principl. of Equiv.

(1) Gravity as scalar field: Particle EOM $\lambda \propto m$ by principle of Eq.
⇒ ds^2 gets modified ⇒ Symmetries of spacetime get modified
(2) $\frac{\partial V}{\partial \phi} = -\Box\phi$ $V(\phi) \propto \lambda n \phi \propto m n \phi$ $mn = \rho$ ∴ $\Box\phi \propto \rho$
⇒ $\nabla^2\phi \propto \rho$ (Newton's Approx)..
(39) (3) But $E^2 = m^2c^4 + p^2c^2$ $m \xrightarrow{conv.} E$ ∴ ρ is not good option use T^{ab} ∴ $\Box\phi \propto T^a_a$

why Gravity is Not a Scalar field

(4) Vector field: $V^i = (\phi, \vec{V})$ ⇒ $\Box V^i = 4\pi J^i$ ⇒ $\Box\phi = 4\pi\rho$
⇒ $\nabla^2\phi \propto \rho$ ⇒ $\nabla^2\phi \propto T^a_a$ But $T^a_a = 0$ for EM ∴ No couplin
(40) b/w EM & GR ⇒ ∴ Light Doesn't Bend ⇒ Exp. wrong.
(5) Similar for Tensor field. (2)(3)(4) are Exp. facts Only thing which can change is Lorentz Inv.
(6) Principle of Eq. changes the Symmetries of spacetime ⇒ $m_i \frac{d^2x}{dt^2} = -m_g$
⇒ $m_g = m_i$ ⇒ free particle in acc. frame ≡ particle in gravity; PoEq. is not true for EM (λ / m_i). ⇒ By PoEq. line interval ~~should be~~ of acc. fr should be equal to weak static gravity. ← $ds^2 = (1+2gx')dt'^2 - dx'^2$
$ds^2 = e^{\left(\frac{2gx'}{c^2}\right)} ds'^2$ ↳ Rindler metric

(7)

	free Particle	Const acc.
NR	$x = kt$	$x = gt^2/2$
R	$x = v t$	$x^2 - ct^2 = \left(\frac{c^2}{g}\right)^2$

→ $x - t = \frac{c^2}{g} e^{t' - x'}$
$x + t = \frac{c^2}{g} e^{t' + x'}$

(8) Photons up & Down Arg. ⇒ $\Delta t_{down} = \Delta t_{up}\left(1 - \frac{gL}{c^2}\right)$ $ds^2 = (1+2\phi)dt'^2$ ←
$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$
$g_{00} = (1+2\phi)$ (from Both Arguments)
converting into ϕ
(41) Physical Length $= f(\theta) d\phi^2$ Compatibility? $(\Delta t)_x = \Delta t\left(1 + \frac{2\phi}{c^2}\right)$ ↑ ($g=0$)
Similarly
(42) Physical Time $= f(x)\Delta t$ → $ds^2 = g_{ab} dx^a dx^b$ ← plane sh of pap

(9) $d\ell = e^{gx} dx$ in Rindler metric ⇒ $ds^2 = g^2\ell^2 dt'^2 - d\ell^2$ ⇒ $i\tau g = \theta$, $\ell \equiv r$

(43)

(38) Po

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particle Coupled to Scalar field

(38) Gra → A is L.I → L is L.I. $A = -\int L(x^i, u^i)\, ds$ u^i

① L ① $L(x^i, u^i) = -(+m + \lambda\phi(x^i) + q A_i(x^j) u^i + A_{ij}(x^k) u^i u^j)$

① G

⇒ $L(x^i, u^i) = -m - \lambda\phi(x^i)$

② ② $A = -m\int ds - \lambda\int \phi\, ds - q\int A_i u^i ds - \int A_{ij}\frac{dx^i}{d}$

③ $A = -m\int \frac{dt}{\gamma} - \lambda\int \frac{\phi}{\gamma} dt \Rightarrow L = -\frac{m}{\gamma} - \frac{\lambda\phi}{\gamma}$ as $A = \int L$

(39) ③ $\frac{dE}{dt} = -\lambda \frac{\partial\phi(x^i,t)}{\partial t}\gamma$ ← $L_{NR} = -mc^2 + mv^2/2 - \lambda\phi(x)\gamma$ → just as $L = \frac{mv^2}{2}$

④ $\delta A = -\int d(m^* U_i \delta x^i) + \int \frac{d\epsilon}{d\epsilon}\left(\frac{d(m^* U_i)}{ds} - \lambda\partial_i\phi\right) ds\, \delta x^i$

④

⑤ $\frac{du^i}{ds} = \frac{\lambda\partial^i\phi}{m+\lambda\phi} - \frac{\lambda\partial_j\phi\, u^j u^i}{m+\lambda\phi}$ (force dep. on vel.)

(40) in NR $\phi_{,i}$ goes away

⑤ $\frac{d\vec{p}}{dt} = -\lambda\vec{\nabla}\phi$ in NR lt.

Canonical mom. picks up field dep. term.

⑥ $\frac{\partial L}{\partial\vec{v}} = \vec{p} = (m+\lambda\phi)\gamma\vec{v} = m^*\gamma v$ → $E^2 = p^2 + m^2$

$H = p\dot{q} - L = m^*\gamma v^2 + \frac{m^*}{\gamma} = m^*\gamma = (m+\lambda\phi)\gamma$

⑦ $\frac{\delta A_c}{\delta x^i} = -m^* U_i = -P_i = (-E, \vec{p}) \Rightarrow \frac{\delta A_c}{\delta x^0} + E = 0$

$\frac{\delta A_c}{\delta\vec{x}^b} = \vec{p}$

⑧ Def. $P_i = m^* U_i$ = 4 mom.

$P_i = (m^*\gamma, -m^*\gamma\vec{v}) = (E, -\vec{p})$ ⑨ Rel. H-J Eqn $m^{*2} = \eta^{ab}\partial_a$

Dynamics of Scalar field

① $A = -m\int ds - \lambda\int\phi\, ds - \int L_f d^4x = -m\int ds - \lambda\int n\phi\, d^4x - \int L_f d^4x$

(41) $n(x^i) = \int_{-\infty}^{\infty} ds\ \delta_D(x^i - z^i)$ = Particle Density. ext. specified

② $\delta A = -\lambda\int \partial_i(\pi^i\delta\phi) d^4x + \lambda\int\left(\partial_i\pi^i - \frac{\partial L_T}{\partial\phi}\right)\delta\phi\, d^4x$ $L_T = (L_f + n\phi)$

$\partial_i\pi^i = \frac{\partial L_t}{\partial\phi}$ $L_f(\phi, \partial_a\phi) = \partial_i\phi\partial^i\phi - U(\phi)$ = Lorentz inv. = $L(v^2)$

||| $L_t(\phi, \partial_a\phi) = \frac{\partial_i\phi\partial^i\phi}{2} - V(\phi) = \frac{\dot{\phi}^2}{2} - V$ → ϕ is L.I. is assu

$\Box\phi = -\frac{\partial V}{\partial\phi}$ $\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$ $\frac{\partial L}{\partial\vec{v}} = p$

③ $\frac{\delta A_c}{\delta\phi} = -\pi^0 = -p^0$ Def. $p^a = \pi^a$ = can. mom. $= \partial^a\phi = (\dot{\phi}$

(43) ④ $T^a{}_b = \pi^a\partial_b\phi - \delta^a_b L_t(\phi, \partial_i\phi) \equiv p\dot{q} - L(x^i, u^i)$

$T = \partial^a\partial_b\phi - \delta^a_b L_t$ = Symm. Tensor

$T^0{}_0 = \frac{\dot{\phi}^2}{2} + \frac{|\vec{\nabla}\phi|}{2} + V$ = Energy Density ⑤ in CM if $L(t)$ then $\frac{dH}{dt} = 0$; Here if $L(x^i)$ $\partial_a T^a{}_b = 0$

Def $\int d^3x\, T^0_? = P_?$

L-7

1) $g_{00} = (1 + 2\phi)$ $c = 1$

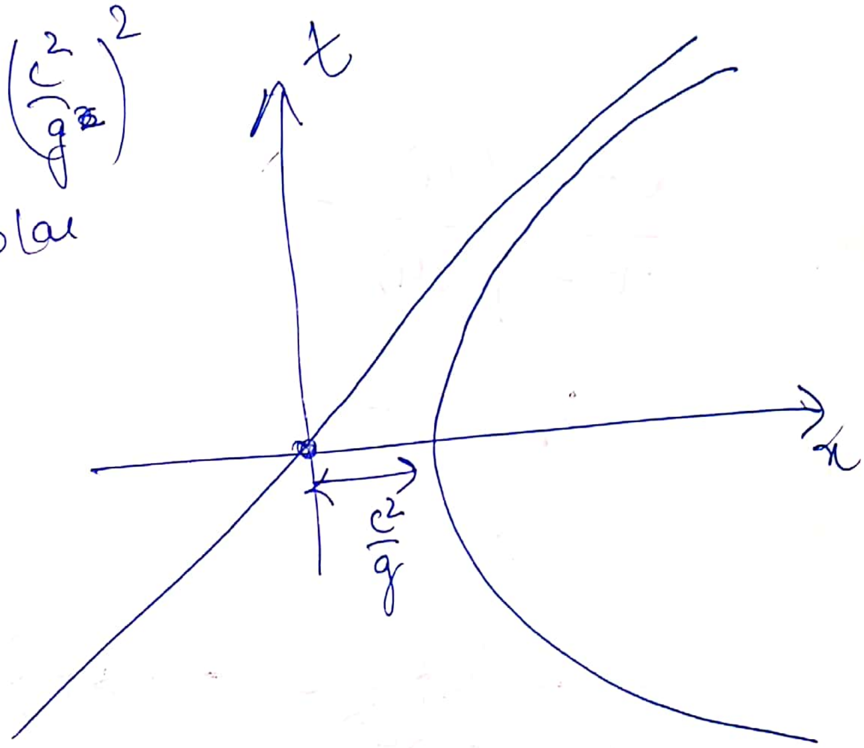
2) Traj of uniformly acc. observer w.r.t. Inertial frame

$$X = \frac{c^2 \cosh g\tau/c}{g}$$

$$cT = \frac{c^2 \sinh g\tau/c}{g}.$$

$$X^2 - cT^2 = \left(\frac{c^2}{g}\right)^2$$

∴ Hyperbolae

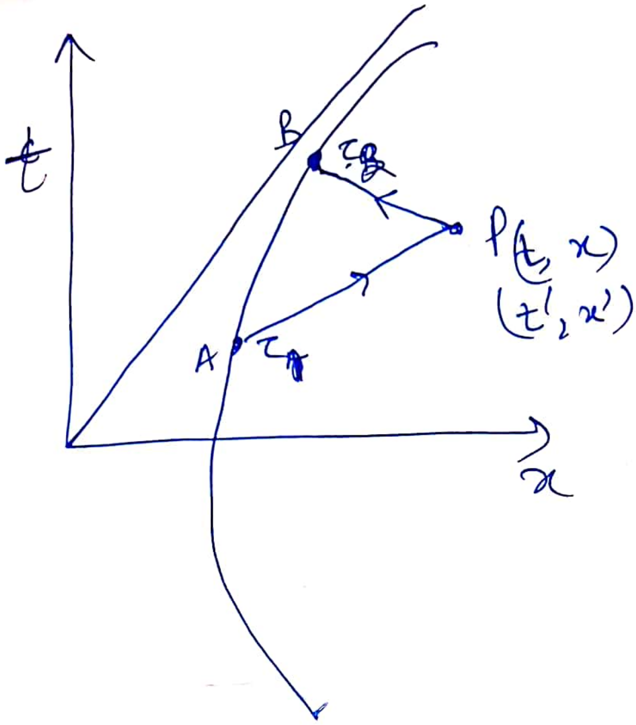


3) $\frac{c^2}{g} = [L]$ Dimension.

$\frac{c^2}{g_{Earth}} = 1$ lt yr $= f(\text{Mass}_{Sun}, \text{Dist}_{B/w\ E\&S})$

$g_{Earth} = f(\text{mass}_{Earth}, \text{Rad}_{Earth})$

(4)



$P(t, x)$

$$\left.\begin{aligned} t' &= \left(\frac{\tau_2+\tau_1}{2}\right) \\ x' &= \left(\frac{\tau_2-\tau_1}{2}\right)c \end{aligned}\right\} \circledast$$

$A : \left(f_0(\tau_1), f_1(\tau_1)\right)$

$B : \left(f_0(\tau_2), f_1(\tau_2)\right)$

$$\frac{x - f_1(\tau_1)}{t - f_0(\tau_1)} = c$$

$$x^2 = \left(\frac{c^2}{g}\right)^2 + $$

$$\frac{f_1(\tau_2) - x}{f_0(\tau_2) - t} = -c$$

$$f_1^2(\tau) = c^2 f_0^2(\tau)$$

$$x - c f_0^2(\tau_1) = ct - c f_0(\tau_1)$$
$$- (c^2/g)^2$$

$$c f_0^2(\tau_2) + \left(\frac{c^2}{g}\right)^2 - x = c\, t - c f_0(\tau_2)$$

$$c f_0^2(\tau_2) + c f_0(\tau_2)$$

$$x - \sqrt{c^2 f_0^2(\tau_1) + \left(\frac{c^2}{g}\right)^2} = ct - c f_0(\tau_1)$$

$$\sqrt{c^2 f_0^2(\tau_2) + \left(\frac{c^2}{g}\right)^2} - x = ct - c f_0(\tau_2)$$

~~$$d\tau = \int \frac{dt}{x} = \int dt \sqrt{1 - \frac{1}{x^2}}$$~~

~~From (1) $2xv = 2c$~~

~~$v = \frac{c}{x}$~~

~~$$d\tau = \int \frac{dt}{x} \sqrt{x^2 - 1} = \int \frac{dt\left(\left(\frac{c^2}{g}\right)^2 + ct^2 - 1\right)}{\sqrt{ct^2 + \left(\frac{c^2}{g}\right)^2}}$$~~

~~$$= \int dt$$~~

$$ct = \frac{c^2 \sinh g\tau}{g}$$

$$\boxed{t = \frac{c}{g} \sinh g\tau} \Rightarrow f_0(\tau_1) = \frac{c}{g} \sinh g\tau_1/c$$

$$f_1(\tau) = \frac{c^2 \cosh g\tau/c}{g}$$

From (*) $t' + \frac{x'}{c} = \tau_2$ $\quad t' - x'/c = \tau_1$

~~$$\frac{t'}{c} + \frac{x'}{c} = \tau_2 \qquad \frac{t'}{c} - \frac{x'}{c} = \tau_1$$~~

$$x - \sqrt{\frac{c^4}{g^2} \sinh^2 g\tau_1 + \frac{c^4}{g^2}} = ct - \frac{c^2}{g} \sinh g\tau_1$$

$$x - ct = \sqrt{\frac{c^4}{g^2} \sinh^2 \frac{g}{c}\left(\frac{t' - x'}{c}\right) + \frac{c^4}{g^2}} - \frac{c^2}{g} \sinh \frac{g}{c}\left(\frac{t' - x'}{c}\right)$$

$$x + ct = \sqrt{\frac{c^4}{g^2} \sinh^2 \frac{g}{c}\left(\frac{t' + x'}{c}\right) + \frac{c^4}{g^2}} + \frac{c^2}{g} \sinh \frac{g}{c}\left(\frac{t' + x'}{c}\right)$$

$\sqrt{\sinh^2(x)+1} = \cosh(x)$

& $\cosh x - \sinh x = e^{-x}$

$\therefore$

$$x - ct = \frac{c^2}{g}\left(\exp\left(-\frac{g}{c}\frac{(t'-x')}{c}\right)\right) \quad \text{—(1)}$$

$$x + ct = \frac{c^2}{g}\left(\exp\left(+\frac{g}{c}\frac{(t'+x')}{c}\right)\right)$$

(5) $dx - c\,dt = -\exp\left(-\frac{g}{c}\frac{(ct'-x')}{c}\right)(dct' - dx')$

$dx + c\,dt = \exp\left(g\frac{(ct'+x')}{c^2}\right)(dct' + dx')$

$dx^2 - c^2dt^2 = -(c^2dt'^2 - dx'^2)\exp\left(\frac{2gx'}{c^2}\right)$

$dx^2 - c^2dt^2 = (dx'^2 - c^2dt'^2)\exp\left(\frac{2gx'}{c^2}\right)$

$\therefore$ $$dx^2 - c^2dt^2 = \exp\left(\frac{2gx'}{c^2}\right)(dx'^2 - c^2dt'^2)$$

$\therefore$ The metric gets modified in acc. frame which is not the case in SR.

(6) from (1)

~~$x = \frac{c^2}{g}e^{\frac{gx'}{c^2}}\sinh\left(\frac{gt'}{c^2}\right)$~~

~~$t = \frac{c}{g}e^{\frac{gx'}{c^2}}\cosh\left(\frac{gt'}{c^2}\right)$~~

(6) from (1)

$$x = \frac{c^2}{g} e^{gx'/c^2} \cosh\left(\frac{gt'}{c}\right)$$

$$t = \frac{c^2}{g} e^{gx'/c^2} \sinh\left(\frac{gt'}{c}\right)$$

(2)

The transfn is non linear & hence do not preserve ds^2

(7) As in Lorentz Transfn

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$x' = 0$ then $t' = \tau$

$x = \gamma v \tau$ & $x = vt$

$\therefore t = \gamma\tau$ as verified.

(8) in (2) $x' = 0$ $t' = \tau$

(*) $$\left[\begin{aligned} x &= \frac{c^2}{g} \cosh\left(\frac{g\tau}{c}\right) \\ t &= \frac{c^2}{g} \sinh\left(\frac{g\tau}{c}\right) \end{aligned}\right\}$$ which is equivalent to [2]

(9) Derive (*) from ~~$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}$~~

as $u^i u_i = -1$

$\therefore u^{0\,2} - u^{1\,2} = -1$

gen. soln for this

$u^0 = \cosh f(\tau)$

$u^1 = \sinh f(\tau)$

$a^0 = \frac{df(\tau)}{d\tau} \sinh f(\tau)$

$a^1 = \frac{df(\tau)}{d\tau} \cosh f(\tau)$

$$\|a\| = \sqrt{a^i a_i} = \dot{f}\sqrt{\sinh^2 f(\tau) - \cosh^2 f(\tau)}$$

$$= \dot{f}\sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2}$$

$$= \dot{f}\sqrt{\frac{2+2}{4}}$$

$$\|a\| = \dot{f}(\tau)$$

for Uniform acc.

$$\therefore \quad f(\tau) = g\tau$$

$$\therefore \quad u = (\cosh g\tau, \sinh g\tau, 0, 0)$$

$$\frac{dx^0(\tau)}{d\tau} = \cosh(g\tau)$$

$$x^0(\tau) = \frac{1}{g}\sinh(g\tau)$$

$$x^1(\tau) = \frac{1}{g}\cosh(g\tau)$$

→ Static

$$ds^2 = c^2dt^2 - dx^2 + dy^2 - dz^2 = e^{\left(\frac{2gx'}{c^2}\right)}(c^2dt'^2 - dx'^2) + dy^2 + dz^2$$

ρ & τ are our new coordinates $\left\{ \begin{array}{l} x = \rho \cosh\tau \\ t = \rho \sinh\tau \end{array} \right\}$ $\rho \geq 0$

$$dt^2 - dx^2 = \rho^2 d\tau^2 - d\rho^2$$

$$\text{Diag} = (\rho^2, -1, -1, -1) \rightarrow \text{Static}$$

(10) from (5)

$$(c^2 dt^2 - dx^2) = \exp\left(\frac{2gx'}{c^2}\right)(c^2 dt'^2 - dx'^2)$$

$$\approx \left(1 + \frac{2gx'^2}{c^2}\right)(c^2 dt'^2 - dx'^2)$$

$$\approx c^2 dt'^2 - dx'^2 + 2gx'^2 dt'^2$$

$$ds^2 \approx \left(1 + \frac{2gx'}{c^2}\right) c^2 dt'^2 - dx'^2$$

$$\approx \left(1 + \frac{2\phi}{c^2}\right) c^2 dt'^2 - dx'^2$$

Transformation to non inertial frames always lead to line interval $ds^2 = g_{ik}(t, \vec{x}) dx^i dx^k$

Here it only depends on x as Uniform acc. is considered

See L-(6) (57)

In (57) we found that gravity changes the metric.

from (10) This metric is same to gravity.

$\therefore$ In acc. frame observer can't disting. b/w gravity & acc locally.

$\Rightarrow$ which Implies Principle of Equivalence.

~~New coordinate~~ $c = 1$

(11) Let $dl = e^{gx} dx$

$$l = \frac{e^{gx}}{g}$$

we want $l = 0$ when $x = 0$

$\therefore l_g = l - 1/g$

$$ds^2 = g^2 l^2 dt'^2 - dl^2$$

$$= (1 + l_g g)^2 dt'^2 - dl_g^2$$

(12) When $ds^2 = \exp\left(\frac{2gx'}{c^2}\right)(c^2 dt'^2 - dx'^2)$

When $g = 0$ this is flat spacetime.

In $ds^2 = g^2 l^2 dt'^2 - dl'$ it is not so apparent

$\therefore \; l_l = l - 1/g \Rightarrow l_g = lg - 1$

$l_g = e^{gx} - 1$

$ds^2 = (1 + l_g)^2 dt'^2 - dl_l^2$

$\lim_{g \to 0} l_g = 0$

$\therefore \; ds^2 = dt'^2 - dx'^2$

flat spacetime.

(13) **Rindler metric**

$$ds^2 = \exp\left(\frac{2gx'}{c^2}\right)\left(c^2 dt^2 - dx^2\right)$$

(14) from (11)

$$ds^2 = g^2 \ell^2 dt^2 - d\ell^2$$

$$-ds^2 = -g^2 \ell^2 dt^2 + d\ell^2$$

let $t \longrightarrow i\tau$

$$-ds^2 = g^2 \ell^2 \, d(i\tau)^2 + d\ell^2$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

$\therefore$ Here $\ell \equiv r$

$i\tau g \equiv \theta$

$$\therefore \; -ds^2 = r^2 d\theta^2 + dr^2$$

Plane sheet of Paper in Polar Coordinate. $\therefore$ Rindler Metric is a particular type of metric

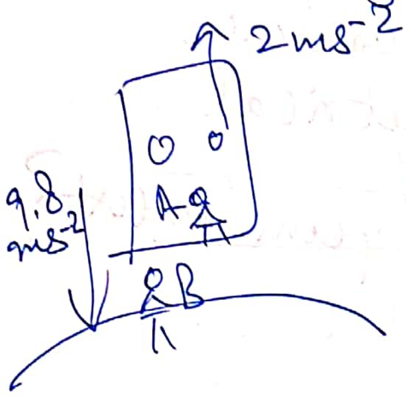
(15) θ is coordinate with Period $[0, 2\pi]$

$\therefore$ Period of $i\tau$ is $\left[0, \frac{2\pi}{g}\right]$

This shows relationship B/w QFT & TD.

Condense matter course $\rightarrow$ Periodicity in imaginary time $\equiv$ Temp.

(16) If the Box is near Earth & also accelerating.



Net acc. of Balls = 11.8 ms^{-2}

we don't know how much of it is due to acc. of Box & how much of it is due to gravity?

for B I know ball would fall at $a_1 + a_2$ out of which a_2 is of box & a_1 is gravity.

But for A ball would fall at $a_1 + a_2$ but he doesn't know if all of it is g, all of it acc. or mixture of both.

(17) ∴ We have lost our ability to distinguish B/w effects of coordinate transformation from acc. to inertial frame from Real gravity. The presence of weak grav. tells us we should use same coordin. transf as acc. in both cases are same As metric

(18) The presence of gravity tells that we can change coordinates in arbitrary manner. how?

Imp

$$x' = f_1(x,v,t)\,x + f_2(x,v,t)\,t$$
$$t' = g_1(x,v,t)\,x + g_2(x,v,t)\,t$$

for gravity arbitrary we can't find transfn

Which is what we did in acc.

Relation B/w

& $x' = f_1(x,v,t)\,x + f_2(x,v,t)\,t$.

(19) General Covariance

All the laws should be invariant under any coordinate transfn

General Covariance in Geometrical Context?

SR Post | in Geom. Context?

$x^a \longrightarrow x^{a'}$

$x^{a'} = f^a(x^a)$

(20) $\vec{F} = m\vec{a}$ invariant under Rotations.

Maxwells eqn Remain inv. under Lorentz Rotation.

Now any law should be inv. under any arbitrary Transformation.

(21) let
$dl^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2$ — (1)

$dl^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2$ — (2)

(1) — if $x^1 = r \in [0, \infty)$, $x^2 = \theta \in [0, 2\pi]$ } Plane in Polar coord.

(2) — if $x^1 = \theta$, $x^2 = \varphi$ } Sphere with fixed R.

(22) I know (1) is flat coz I can write down a coordinate transfn $r\sin\theta = x$, $r\cos\theta = y$

s.t. $dl^2 = dx^2 + dy^2$ ∴ flat

∴ a coordinate transfn $x = f(x^1, x^2)$
$y = g(x^1, x^2)$

s.t. $dl^2 = dx^2 + dy^2$.

In ② case we can't do this
∴ Curved Surface

(23) Now $ds^2 = g_{ab}(t,\vec{x})\, dx^a dx^b$

Now here there is no coordinate Transfn which convert this into Lorentzian form.

But if

$$ds^2 = \exp\left(\frac{2gx'}{c^2}\right)\left(c^2 dt'^2 - dx'^2\right)$$

in this case we can convert it into Lorentzian

$$c^2 dt^2 - dx^2$$

by $x' = \frac{c^2}{g} e^{gx/c^2} \cosh\left(\frac{gt}{c}\right)$

$t' = \frac{c^2}{g} e^{gx/c^2} \sinh\left(\frac{gt}{c}\right)$

∴ Some $g_{ab}(t,\vec{x})$ can be reduced & some cannot.

(24) $g_{ab}(t,\vec{x})$ is 10 fn.

if I make $x^a \to x^{a'} = f^a(x^a)$

∴ I have 4 f^0, f^1, f^2, f^3 in my hands.

∴ There is noway I can use freedom of 4 fn & bring back 10 fn to a preassigned form.

∴ In general it cannot be brought to a flat spacetime. → $g_{ij} f^i f^j = 0$ } → 10 conditions & 4 fn

Analogy: $x+y+z=0$ more cond less variable ↓ Not sure soln exists whether

(25) given any pt. x^{ai} we have $g_{ab}(x^i)$

then $x^{i'} = f^i(x^i)$ s.t. $\eta_{ab}(x^i) = g_{ab}(x^i)$

at that pt. Then I want to kill 1st derivatives

& higher derivatives.
& if all derivatives = 0 then it is flat spacetime ? (95)

(26) Let that point be origin
& Taylor Expand around (O).
∴ Origin map to origin.

Taylor Exp: $x'^a = B^a_k x^k + C^a_{ij} x^i x^j + D^a_{ijk} x^i x^j x^k - - -$

free parameters

Taylor Exp. around a

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 - - -$$

(27) Free parameters are used to reduce to η_{ab} the g_{ab}
& as many derivatives of $g_{ab} = 0$ as possible.

(28) $g_{ab}(0) = \eta_{ab}$; 10 conditions

B^a_k : 16 ∴ easily can satisfy 10.

N Dim $ds^2 = g_{ab} dx^a dx^b$ $a = 0, 1, 2 - - - n-1$

(29) $\frac{N^2 - N}{2} + N$ = Total No. of ind. elements in 2 ~~index~~ Rank Tensor g_{ab}

$\frac{N^2 + N}{2} = \frac{N(N+1)}{2}$ = No. of ind. elements in g_{ab}.

B^a_k has N^2 elements

Excess DoF $= N^2 - \frac{N^2 + N}{2} \Rightarrow \frac{N^2 - N}{2}$

$= \frac{N(N-1)}{2} \Rightarrow N_C$

As. L.T. ... g_{ab} ... I am still able

(30) Now the Excess Dof is 6 [coord. transf. I should have exact no. of Dof by which I can do L.T. to do L.T.]

Imp & when we do L.T. we rotate about plane & there are $^4C_2 = {^N}C_2 = 6$

$\therefore$ These 6 Dof is to do L.T.

(31) Demand $\partial_i g_{ab}(0) = 0$

No. of Ind. fn in g_{ab} are $\frac{N(N+1)}{2}$

eg. $f_1(x, y)$, $f_2(x, y)$

$$\frac{\partial f_1(x,y)}{\partial x} = 0$$

$$\frac{\partial f_1(x,y)}{\partial y} = 0$$

$$\frac{\partial f_2(x,y)}{\partial x} = 0$$

$$\frac{\partial f_2(x,y)}{\partial y} = 0$$

$\therefore$ 4

Each multiplied by 2

Now in $\partial_i g_{ab}(0) = 0$

$$N \times \frac{N(N+1)}{2} = \text{Tot. No. of conditions}$$

$$\Rightarrow \frac{N^2(N+1)}{2}$$

(32) & we have c^a_{ij} symmetric $\equiv N \times {^{N+1}}C_2$

$\therefore$ we have $\frac{N^3 - N}{2} + N = \frac{N^2(N+1)}{2}$ ind. elements

$\therefore$ This is exact to total No. of cond.

$\therefore$ we can satisfy.

(33) $\therefore$ We can always do coordinate trans. & in which $g_{ab}(x^i_1) = \eta_{ab}(x^i_1)$

& $\partial_i g_{ab}(x^i_1) = 0$

Guess

No. of ind. comp. $g_{ab} = {}^{N+1}C_2 = \frac{N(N+1)}{2}$

(34) $\partial_i \partial_j g^{(0)}_{ab} = 0$ — symmetric, symmetric. Both sym.

No. of ind. comp $\partial_i \partial_j = {}^{N+1}C_2$

$\therefore$ No. of ind comp $\partial_i \partial_j g_{ab} = ({}^{N+1}C_2)^2$

$$\frac{N(N+1)}{2} \times \frac{N(N+1)}{2} = \frac{(N(N+1))^2}{4}$$

~~= Total No. of cond. using D^a_{ijk} symmetric~~

~~$\therefore \frac{N^4 - N}{2} + N = \frac{N(N^3+1)}{2}$~~

Tot. No. of cond. using $D^a_{ijk} = N \, {}^{N+2}C_3$

Excess D.o.f $= \frac{(N(N+1))^2}{4} - \frac{N^2(N+2)(N+1)}{6}$

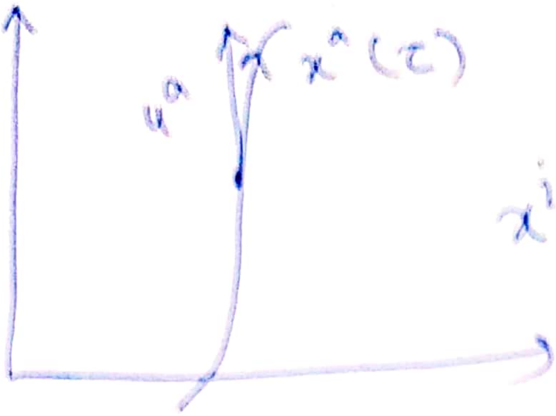
$$= N^2(N+1)\left[\frac{N+1}{4} - \frac{N+2}{6}\right]$$

$$= \frac{N^2(N+1)(N-1)}{12}$$

(35) Completely symmetric tensor with s indices in N dim = ${}^{N+S-1}C_S$

Scanned with CamScanner

(36)



$$x^{i'} = f^{i'}(x^i)$$

$$\boxed{\frac{dx^{a'}}{d\tau} = \frac{\partial x^{a'}}{\partial x^{b}} \frac{dx^b}{d\tau}}$$

Coordinate Transf^n in Arbitrary Coordinate.

$$u^{a'} = \frac{\partial x^{a'}}{\partial x^b} u^b$$

let $T^{ab} = u^a v^b$

$$u^{a'} v^{b'} = T^{a'b'} = \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^{b'}}{\partial x^b} u^a v^b$$

$$\boxed{T^{a'b'} = \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^{b'}}{\partial x^b} T^{ab}}$$

Prime above → as in L.T.

(37) In Lorentz transf^n

$$\frac{\partial x^{a'}}{\partial x^b} = L^{a'}_{\ b}$$

$$\therefore \quad dx^{a'} = L^{a'}_{\ b}\, dx^b.$$

(37) x^i is not 4-vector

dx^i is 4 vector.

(38) $\partial_i \phi = \frac{\partial \phi}{\partial x^i} = \frac{\partial \phi'(x^{i'})}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^i}$

$$\boxed{\partial_i \phi = \left(\frac{\partial x^{i'}}{\partial x^i}\right) \partial_{i'} \phi'} \Rightarrow \partial_i \phi = L^{i'}_i \partial_{i'} \phi$$

$$\partial_{i'} \phi' = \frac{\partial \phi}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} = \left(\frac{\partial x^i}{\partial x^{i'}}\right) \partial_i \phi$$

$\Downarrow$

msR

$$\partial_{i'} \phi = L^i_{i'} \partial_i \phi$$

(39) Th. $A_i B^i = g(\vec{A}, \vec{B})$ Remains inv. under arbitrary Transfn - as it is a scalar.

Proof: $A_{i'} B^{i'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^j} A_i B^j$

$= \frac{\partial x^i}{\partial x^j} A_i B^j = \delta^i_k A_i B^j$

$A_{i'} B^{i'} = A_i B^i$

$\therefore g(\vec{A}, \vec{B})$ Remains invariant under arbitrary Transf.

(40) Def $A_j = \eta_{jk} A^k$ (MSR)

Def. $A_j = g_{jk} A^k$

(41) $g(\vec{A}, \vec{B}) = A^i B_i$

$g(d\vec{x}, d\vec{x}) = dx^i dx_i$

$= g_{ij}\, dx^i dx^j = ds^2$

(42) as $g(\vec{A}, \vec{B})$ remains inv.

$\therefore$ $g(d\vec{x}, d\vec{x}) = ds^2$ would remain inv. under arbitrary Transf^{n}

$ds^2 = g_{ab}\, dx^a dx^b = g_{a'b'}\, dx^{a'} dx^{b'}$

which can be seen manually also

$g_{ab}\, dx^a dx^b = g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} dx^{a'} dx^{b'}$

$g_{a'b'}$ $\longrightarrow$ $\therefore$ g_{ab} is a tensor

~~by SR η_{ab} is prod to be~~ Tensor as we know from

(43) $d^4x \xrightarrow{L.T} d^4x'$ (similar to η_{ab})

$\Rightarrow d^4x = d^4x'$ in (LT)

$d^4x' = \left|\frac{\partial x^{i'}}{\partial x^j}\right| d^4x = J d^4x$ — (1)

(44) $g' = J^{-2} g$

as det g_{ab} is $-$ve

as η_{ab} is spcl case of g_{ab}

$\therefore$ $-g' = -gJ^{-2}$ $\Rightarrow$ $\sqrt{-g'} = \frac{\sqrt{-g}}{J}$

$\therefore$ Putting in ① $d^4x' = \frac{\sqrt{-g}}{\sqrt{g'}} d^4x$

$\therefore \sqrt{-g'}\, d^4x' = \sqrt{g}\, d^4x$

$\therefore$ this Volume invariant remains invariant.

~~(45) Th: ln (det M) = Tr (ln M)~~

(45) $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

$g_{a'b'}$ = Metric = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ as $dl^2 = dx^2 + dy^2 + dz^2$

$dl^2 = g_{ab}\, dx^a dx^b$

$|g_{a'b'}| = g' = r^4 \sin^2\theta$

$\therefore \sqrt{g}\, d^3x = \sqrt{g'}\, d^3x'$

$\sqrt{g}\, dx\, dy\, dz = \sqrt{g'}\, dr\, d\theta\, d\phi$

$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

this is Jacobian.

(46) Another way to obtain

(47) Derivative of scalar is 4 covector

But in GR, $\partial_{i'} v^{j'}$ do not transform as tensor under general coordinate transf. though they do in L.T.

(48) $r^2 \sin\theta \, d\theta \, d\phi \, dr$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

↑ metric

$\ln M$ = Taylor Exp.

(48) Th.

$$\ln(\det M) = \mathrm{Tr}(\ln M)$$

Proof: → Diagonalize the matrix

→ Determinant = Prod. of Eigenvalues $= \lambda_1 \lambda_2 \cdots \lambda_n$

$$\ln(\lambda_1 \lambda_2 \cdots \lambda_n) = \sum_i \ln \lambda_i$$

→ Eigen values of $\ln M$

$\Rightarrow \ln \lambda_1, \ln \lambda_2, \cdots \ln \lambda_n$

$$\therefore \ln(\det M) = \mathrm{Tr}(\ln M)$$

(49) $\delta \ln(\det M) = \delta \, \mathrm{Tr}(\ln M)$

$$\frac{\delta(\det M)}{\det M} = \mathrm{Tr}(M^{-1} \delta M)$$

δ & Tr Commute as Tr is just the sum.

Inverse

(50) $\eta_{i'j'} \eta^{j'l'} = \delta_{i'}^{l'}$

Inverse

$g^{i'k'} g_{k'l'} = \delta^{i'}_{l'}$

(51) $\eta^{j'l'}$ is the inverse

Now

$$\eta_{i'j'}\eta^{j'l'} = \delta^{l'}_{i'}$$

'Kronecker Remains invariant

$$L^i_{i'} L^j_{j'} \eta^{j'l'} \eta_{ij} = \delta^i_l$$

$$\eta^{j'l'} = L^{i'}_i L^{j'}_j \eta^{jl} \delta^i_l$$

$$\boxed{\eta^{j'l'} = L^{i'}_l L^{j'}_j \eta^{jl}}$$

which is consistent.

(52) Similarly for $g^{i'k'} g_{k'l'} = \delta^{i'}_{l'}$

(53) Using (49) let $g_{ab} = M$ then $M^{-1} = g^{bc}$

$$\therefore \frac{\delta g}{g} = Tr(g^{bc}\delta g_{ab})$$

as g_{ab} is sym $\therefore g_{ab} = g_{ba}$

$$\frac{\delta g}{g} = Tr(g^{bc}\delta g_{ab})$$

$$= g^{bc}\delta g_{bc} \quad \text{(as Trace is just Sum of Diag)}$$

$$\delta g = g g^{ab}\delta g_{ab} = -g g_{ab}\delta g^{ab}$$

(54) $\Rightarrow$ $\boxed{\partial_i g = g g^{ab}\partial_i g_{ab}}$

↑ as $\partial_a g^{ik} g_{ik} = 0$

$\therefore g g^{ab} \delta g_{ab}$

(38) using (...)

$$g = r^4 \sin^2\theta$$

~~$\frac{\partial g}{\partial r} = 4r^3 \sin^2\theta = r^4 \sin^2\theta \left(g^{aa}\partial_i g_{aa}\right)$~~
~~$= r^4 \sin^2\theta \left(2r^3 + 2r^3 \sin^2\theta\right)$~~

$$\frac{\partial g}{\partial r} = 4r^3 \sin^2\theta$$

$$\frac{\partial g}{\partial r} = g\, g^{ab}\, \partial_i g_{ab}.$$

$$g^{ab} = A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{r^2\sin^2\theta}\begin{pmatrix} r^2\sin^2\theta & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

$$\frac{\partial g}{\partial r} = \frac{r^4 \sin^2\theta}{r^2\sin^2\theta}\left(g^{aa}\partial_i g_{aa}\right)$$

$$= r^2\left(0 + 2r\sin^2\theta + 2r\sin^2\theta\right)$$

$$\Rightarrow 4r^3 \sin^2\theta$$

LHS = RHS

(SR)

① $v^{i'} = \left(\frac{\partial x^{i'}}{\partial x^j}\right) v^j$

↑ fn of x^j

∴ vector transfn depends on x^i which was not the case in SR

② $\partial_i v^j$ is not a Tensor in GR

This was tensor in SR

③ Derivative means $\lim \frac{v^j(x^i_1) - v^j(x^i_2)}{x^i_2 - x^i_1}$

④ $A^i(x^a_1) + B^i(x^a_1) = C^i(x^a_1)$ } hold in Curved Spacetime as this is followed

$C^{i'}(x^a_1) = \frac{\partial x^{i'}}{\partial x^i} C^i(x^a_1)$ ←

⑤ Similar for multiplication

$T^{ij}(x^a_1) = A^i(x^a_1) B^j(x^a_1)$] This holds in Curved ST as this is followed

$T^{i'j'}(x^a_1) = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} T^{ij}$ ←

ee ric ... ⑩, ⑪

⑥ If ④ & ⑤ holds in Curved spacetime then contraction is also hold.

∴ when going to Non we didn't fall into trouble

⑦ But in Derivative we compute at two diff points.

⑧ $A = \int ds = \int \sqrt{\eta_{ab} dx^a dx^b}$

$\Rightarrow$ EOM

$$\frac{du^a}{d\tau} = 0 \quad \forall a \quad — (1)$$

$\sum (\vec{v} \cdot \vec{\nabla} v^a) e_a$ → Acc. in a^{th} direction $= 0$ $\forall a$

$$\frac{dx^b}{ds} \frac{\partial u^a}{\partial x^b} = u^b \partial_b u^a = 0$$

$v_x \frac{\partial f}{\partial x} + \dots = \frac{df}{d\tau}$

Geometrical meaning : How tangent vector is changing in the direction it is moving. the direction of u^a is 0

Change in u^a along the direction of u^a is 0

⑨ eg. → $(\vec{\nabla} f) \cdot (d\vec{x}) = df = 0$

$df = (\partial_i f) dx^i = 0$

→ $(\vec{\nabla} A) \cdot d\vec{x} = dA = 0$

$(\partial_i A^i) dx^i = 0$

change in A^i along dx^i is 0

⑩ But as change in u^a along tangent vector is 0. This means

∴ Particle is moving in a straight line with constant velocity.

⑪ $\frac{du^{a'}}{ds} = 0$

$L^{a'}_a \frac{du^a}{ds} = 0$

$\Rightarrow \frac{du^a}{ds} = 0$

∴ EOM Remains inv.

→ But this can also be seen as

A is L.I.

∴ EOM has to be L.I.

(12) $\partial_b u^a = \frac{\partial x^j}{\partial x^{b'}} \frac{\partial}{\partial x^j} \left\{ \frac{\partial x^{a'}}{\partial x^k} u^k \right\}$

$= \frac{\partial x^j}{\partial x^{b'}} \frac{\partial x^{a'}}{\partial x^k} \partial_j u^k + \frac{\partial x^j}{\partial x^{b'}} \frac{\partial^2 x^{a'}}{\partial x^j \partial x^k} u^k$

(13) $A = \int ds = \int \sqrt{g_{ab} dx^a dx^b}$

$\delta A = 0$ & B. T $= 0$

$$\frac{du_c}{ds} = \frac{\partial_c g_{ab}}{2} u^a u^b$$

(14) if $g_{ab} = \eta_{ab}$

$\frac{du_c}{ds} = 0$ SR

(15) if $c = 0$ & for metric time ind

$\frac{du_0}{ds} = 0$

$\therefore u_0$ is conserved

(16) if metric is ind. of spatial coordinate then the corresponding momentum is conserved.

$$\frac{du_c}{ds} = \frac{\partial_c g_{ab}}{2} u^a u^b \quad \left(\frac{\partial_c g_{ab}}{2} \to 0\right)$$

(17) $A = \int ds = \int \sqrt{\ }$

$g_{00} = (1+2\phi)$

Imp

in SR $\frac{d^2x^i}{d\tau^2} = 0$ in Cart. $\frac{d^2x}{d\tau^2} = 0$

↑ in Similar Analogy

$\frac{d^2x}{d\tau^2} \neq 0$

as in GR $\frac{d^2x^i}{d\tau^2} \neq 0$

(18) $\frac{du^i}{d\tau} + \Gamma^i_{mn} \underline{u^m u^n} = 0$ →
vel. dep.

Γ involves derivative of metrics

& metrics involve potential $g_{00} = (1+2\phi)$

∴ Γ are like grad. of ϕ. potential.

$a = \Gamma^i_{mn} u^m u^n$

$a = q F_{ik} u^k$

(19) in ED

$\frac{du^i}{d\tau} = q F_{ik} \underline{u^k}$
velo. dep.

F_{ik} are derivatives of vector Potential.

This analogy is not right

(20) Later on

Γ are $\equiv$ vector Potential

derivatives of Γ i.e. Curvature $\equiv F_{ij}$ } Doubt

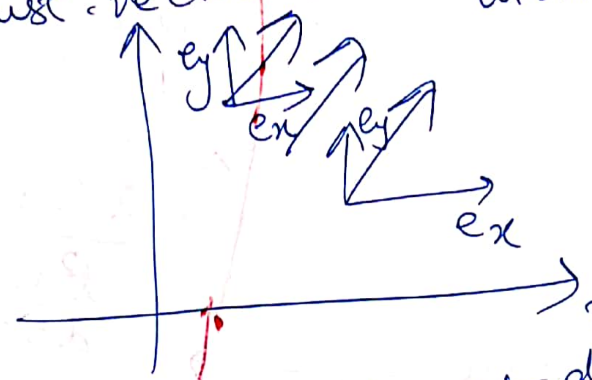
(21) $u^j \partial_j u^k + \Gamma^k_{mj} u^m u^j = 0$

$u^j [\partial_j u^k + \Gamma^k_{mj} u^m] = 0$

$u^j \nabla_j u^k = 0$ } compare with $u^j \partial_j u^i = 0$

(22) $V = v^a e_a$

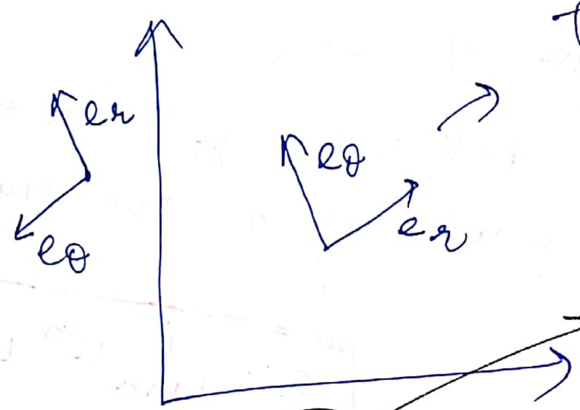
const. vector $\equiv$ Deriv. of comp. of vector w.r.t. coord $= 0$



↑ But important assumption is taken into account ↓

Cartesian Unit vectors do not depend on position

(23)



→ This e_θ, e_r could be due to our coordin. transf. or due to curved surface.

$$\nabla_j \vec{v} = (\partial_j v^a) e_a + v^a \nabla_j e_a$$

$(\partial_j v^a)$ → How the comp. is changing

$\nabla_j e_a$ → How the Basis vectors are changing

(24) $\nabla_j e_a$ is also a vector as they are just shift in Basis vector as it is the lt. of diff. of 2 vectors

Imp

$$\therefore \nabla_j e_a = \beta^b_{aj} e_b$$

→ **Affine Connections**

$$\therefore \nabla_j \vec{v} = \left[(\partial_j v^b) + v^a \beta^b_{aj}\right] e_b$$

Now see similarly $\nabla_j \vec{v}$ as lt. of Diff of 2 vectors being expanded in basis e_b.

Compare with (21)

$$(\nabla_j \vec{v}) = (\nabla_j v^a) e_a$$

→ Component

(25) $\partial_j V + \Gamma_j V = 0$

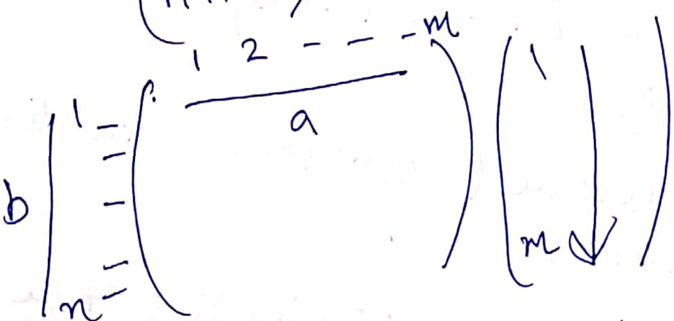
$\Gamma_j V$ ⟶ (matrix) (column vector)

V = column vector

Γ = set of Matrices with 2 indices

(26) $\Gamma^b_{aj} V^a$

$(n \times m)(m \times 1) = (n \times 1)$



$a = 1 - - - - m$

$\Gamma^b_{a0}, \Gamma^b_{a1}, \Gamma^b_{a2}, \Gamma^b_{a3}$ are 4 matrices.

(27) $\partial_j V^a + \Gamma^a_{bj} V^b$ ⟶ Gauge Connection ??

I'll add another V.S.

Gauge Theory

Every pt. in space

if my spacetime is 4D $j = 0, 1, 2, 3$.

At Every pt. I add n dim. space. (Internal space)

Then in internal space vectors have n comp.

$a, b = 1, - - - n$

∴ Index space of index (a, b) should be equal to j.

(28) **Def.** $\nabla_j v^a = \partial_j v^a + \Gamma^a_{bj} v^b$ (Gauge Covariant Derivative)

113.

(29) Now if $\overset{\circ}{\Gamma} \neq 0$ then either we are using coordinates which are making it $\neq 0$ or it is due to genuine curvature.

(30) if it is all due to coordinates then we can do coord. transf. in which $\Gamma = 0$ $\forall$ all.

like Polar coord $\longrightarrow$ Cart. in 2D

But if it is due to genuine curvature then we can't do coord. transf. in which $\Gamma = 0$ $\forall$ all.

(31) We have till now learned covariant Derivative on Contravariant vector.

(32) Def. $\nabla_j \phi = \partial_j \phi$ covariant Derivative of a Scalar.

---

(33) Γ - are derivatives of metric

metric is potential

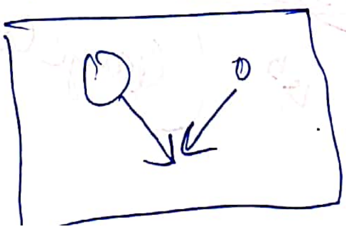
Derivative of pot. is acc.

local acc. ~~E~~ ~~of grav.~~ can be vanished

$\therefore$ Γ can't alone tell if we are ~~on~~ genuine curv. or due to coord. transf.

---

(34) derivative of Γ tells us about curvature.



(35) $\nabla_j (A^k B_k) = \partial_j (A^k B_k)$

Cov. Der. of should obey chain Rule.

$B_k \nabla_j A^k + A^k \nabla_j B_k = (\partial_j A^k) B_k + A^k (\partial_j B_k)$

$B_k (\partial_j A^k + \Gamma^k_{mj} A^m) + A^k \nabla_j B_k = (\partial_j A^k) B_k + A^k \partial_j B_k$

$B_k \Gamma^k_{mj} A^m + A^k \nabla_j B_k = A^k \partial_j B_k$

$B_m \Gamma^m_{kj} A^k + A^k \nabla_j B_k = A^k \partial_j B_k$

$\therefore \boxed{\nabla_j B_k = \partial_j B_k - B_m \Gamma^m_{kj}}$ correction Term.

$v^b \nabla_j g_{ab}$ $\quad B^b \partial_j A_b = 0$ $\quad \partial_j g_{ab}$

(36) Covariant deriv. of Tensor

Def. $\nabla_j K^a{}_b = \partial_j K^a{}_b + \Gamma^a_{mj} K^m{}_b - \Gamma^m_{bj} K^a{}_m$

(37) $\nabla_j v_a = \nabla_j (g_{ab} v^b) = (\nabla_j g_{ab}) v^b + (\nabla_j v^b) g_{ab}$

(38) By ch-4 (109) in same way

$\nabla_j B_k$ is a Tensor

like $\nabla_j B^k$

(39) Now if $\nabla_j B_k$ is Tensor

then like $T_{ab} = g_{bm} T_a{}^m$

$\therefore \nabla_j B_k = g_{ka} \nabla_j B^a$

(40) $\therefore \nabla_j (g_{ab}) V^b = 0$

$\therefore \nabla_j (g_{ab}) = 0$

(41) $\nabla_j (g_{ab}) = \partial_j g_{ab} - \Gamma^m_{jb} g_{am} - \Gamma^m_{aj} g_{mb}$ (42)

$= \partial_j g_{ab} - \Gamma_{a,jb} - \Gamma_{b,aj}$ → symm in a, j

↑ symm in j, b

↓ Connection B/w metric & Γ

$= \partial_j g_{ab} - \frac{1}{2}\left(-\partial_a g_{jb} + \partial_j g_{ba} + \partial_b g_{aj} - \partial_b g_{aj} + \partial_a g_{jb} + \partial_j g_{ab}\right)$

$= \partial_l g_{ab} - \frac{1}{2}\left(2 \partial_j g_{ba}\right)$

$= 0$

① $\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0$

for sphere:
we would get answer: Great circle.
for Polar coord. in 2D:
St. line.

② St. Line: 2 Points on any manifold
Curve of least length connecting these two pt.

To define the length of the curve we need notion of metric.

$\vec{\nabla} f \cdot d\vec{x} = df \Rightarrow$

③ Another Def. of St. line
$u^b \partial_b u^a = 0$ — acc. term in fluid mechanics — $(\vec{v} \cdot \vec{\nabla})\vec{v} = 0$

④ $u^b (\nabla_b u^a) = 0$
↳ This generalizes the notion of st. line

⑤ In Affine manifold we only need Γ to defined
metric is not needed.

⑥ ∴ Geodesic can be found using
① length which requires metric
② Γ in $u^b \nabla_b u^a = 0$ which are given in Affine manifold.

Imp

⑦ Both these definition should match
① least length b/w 2 pts
② Going straight $\vec{\nabla} f \cdot dx = 0$

(8) These two definitions ...
by $\nabla_j(g_{ab}) = 0$

(9)

(10) $$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \quad \text{--- (1)}$$

$s = f(\lambda)$ λ : is the new parameter giving the curve.

$$\frac{d}{ds} = \frac{\partial\lambda}{\partial f}\frac{d}{d\lambda} = \frac{1}{\partial f/\partial\lambda}\frac{d}{d\lambda} = \frac{1}{f'}\frac{d}{d\lambda}$$

$$\therefore \frac{1}{f'}\frac{d}{d\lambda}\left(\frac{1}{f'}\frac{dx^i}{d\lambda}\right) + \frac{\Gamma^i_{kl}}{f'^2}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

$$\frac{1}{f'}\frac{d^2x^i}{d\lambda^2} + \left(\frac{dx^i}{d\lambda}\right)\left(\frac{-1}{f'^2}\right)f'' + \frac{\Gamma^i_{kl}}{f'}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

$$\frac{d^2x^i}{d\lambda^2} - \frac{1}{f'}\left(\frac{dx^i}{d\lambda}\right)f'' + \Gamma^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = \frac{1}{f'}\left(\frac{dx^i}{d\lambda}\right)$$

I/c ਮੁੱਖ ਅਧਿਆਪਕ

ਸਰਕਾਰੀ ਹਾਈ ਸਕੂਲ

ਖਿੱਪਾਂਵਾਲੀ ਤਹਿ. ਫਾਜਿਲਕਾ (ਫਿਰੋਜਪੁਰ)

ਨੰ. ਮਿਤੀ........................

ਵਿਸ਼ਾ........ $\ln(\det M) = \mathrm{Tr}(\ln M)$

ਹਵਾਲਾ..........

(1) $\partial_i g = g\, g^{ab}\, \partial_i g_{ab} = -g\, g_{ab}\, \partial_i g^{ab}$

(2) $\partial_i g^{ak} = -g^{ab} g^{ck} \partial_i g_{bc}$ → Indices raised except over all — ve sign.

(3) $\partial_i (\ln \sqrt{g}) = \dfrac{g^{ab}\, \partial_i g_{ab}}{2}$

(3) $\Gamma^a_{ia} = \dfrac{g^{ak}\, \partial_i g_{ak}}{2} = \partial_i \ln(\sqrt{-g})$.

(4) $g^{ab}\, \Gamma^i_{ab} = -\dfrac{\partial_k (g^{ik} \sqrt{-g})}{\sqrt{-g}}$

(5) $\nabla_i A^i = \dfrac{\partial_i (\sqrt{-g}\, A^i)}{\sqrt{-g}}$ = Divergence of vector field $\nabla_i g^{ab} = 0$

(6) $\nabla_i Q^{ik} = \dfrac{\partial_i (\sqrt{-g}\, Q^{ik})}{\sqrt{-g}}$

(7) $\nabla_a T^a_b = \dfrac{\partial_a (\sqrt{-g}\, T^a_b)}{\sqrt{-g}} - \dfrac{1}{2} (\partial_b g_{au}) T^{au}$.

① $\frac{d}{d\tau}(g_{ib}u^b) = \frac{\partial_i g_{ab}}{2} u^a u^b = \frac{du_i}{d\tau}$

② $\frac{du^i}{d\tau} + \Gamma^i_{mn} u^m u^n = 0$

③ $u^a \nabla_a u^b = 0$

④ $\frac{dx^i}{d\lambda} = u^i \quad \tau = f(\lambda) \quad f' = \frac{\partial f}{\partial \tau}$

$\frac{du^i}{d\lambda} + \Gamma^i_{mn} u^m u^n = \frac{f''}{f'} u^i = f(\tau) u^i$

$u^b \nabla_b u^i = \frac{f''}{f'} u^i$

Manipulation → killing vector

Prop. of R

Affine Parameter

fluid (Energy Mom Density) [f have Dis

Surface

$m\frac{du^i}{d\tau} = \gamma^i q$

∴ RHS is same as Geodesic Eqn ①

let $\frac{dx^i}{d\lambda} = v^i$

$$v^b \nabla_b v^i = \frac{f''}{f'} v^i \quad \text{——} ②$$

We started with Geodesic Eqn and non-reparamet it then got ②

But rep. would not change anything ∴

② should still give me same curve

$$v^b \nabla_b v^i = f(\lambda) v^i$$

is also Geodesic Eqn

in RHS anything $\propto v^i$

would make a Geodesic Eqn.

if $f'' = 0$ then Geodesic Eqn remains inv.

i.e. $s = f(\lambda)$ is linear

(15) $\begin{cases} K^a K_a = 0 \\ K^b \nabla_b K^a = 0 \end{cases}$ Null geodesic $\equiv$ Why?

for material particle

$\begin{cases} K^a K_a = 1 \\ K^b \nabla_b K^a = 0 \end{cases}$ → See ch-5 (31)

(16) We can never use proper time for Null geodesic.

(17) From ch-(4) & 5

$$\delta g = g g^{ab} \delta g_{ab}$$

(18) Rem.

$$\boxed{\partial_i g = g\, g^{ab}\, \partial_i g_{ab}}$$

(19) $\partial_i (g^{ab} g_{bc}) = 0$ → δ^a_c

$$g^{ab} \partial_i g_{bc} + (\partial_i g^{ab}) g_{bc} = 0$$

$$g^{ck} g^{ab} \partial_i g_{bc} = -(\partial_i g^{ab}) g_{bc} g^{ck}$$

$$g^{ab} g^{ck} \partial_i g_{bc} = -(\partial_i g^{ab}) \delta^k_b$$

$$g^{ab} g^{ck} \partial_i g_{bc} = -\partial_i g^{ak}$$

Rem.

$$\boxed{\partial_i g^{ak} = -g^{ab} g^{ck} \partial_i g_{bc}}$$

(20) Γ^a_{ka} from (18)

$$\frac{\partial_i g}{g} = g^{ab} \partial_i g_{ab}$$

As g is $-ve$

$$\therefore \frac{\partial_i (-g)}{(-g)} = g^{ab} \partial_i g_{ab}$$

$$\frac{\partial_i \ln(-g)}{2} = \frac{g^{ab}}{2} \partial_i g_{ab}$$

Rem

$$\boxed{\partial_i \ln \sqrt{-g} = \frac{g^{ab}}{2} \partial_i g_{ab}} \quad — (1)$$

(21) $$\Gamma^a_{ia} = \frac{g^{ak}}{2} \left(-\partial_k g_{ia} + \partial_i g_{ka} + \partial_a g_{ik} \right)$$

sym in a, k

$$= \frac{g^{ak}}{2} \left(\underbrace{-\partial_k g_{ia} + \partial_a g_{ik}}_{\text{Ant. in } a, k} \right) + \frac{g^{ak}}{2} \partial_i g_{ka}$$

$$= 0 + \frac{g^{ak}}{2} \partial_i g_{ka}$$

Rem

$$\boxed{\Gamma^a_{ia} = \frac{g^{ak} \partial_i g_{ka}}{2} = \partial_i (\ln \sqrt{-g})} \quad \text{from (1)}$$

(22) $$g^{ab} \Gamma^i_{ab} = \frac{g^{ab}}{2} g^{ik} \left(-\partial_k g_{ab} + \partial_a g_{bk} + \partial_b g_{ak} \right)$$

As $$\boxed{g^{ab} \partial_a g_{bk} = g^{ba} \partial_b g_{ak} = g^{ab} \partial_b g_{ak}}$$

Interchange

$$\therefore g^{ab} \Gamma^i_{ab} = g^{ab} \frac{g^{ik}}{2} \left(-\partial_k g_{ab} + 2 \partial_a g_{bk} \right)$$

Scanned with CamScanner

24

$$= -\frac{g^{ab} g^{ik}}{2} \partial_k g_{ab} + g^{ab} g^{ik} \partial_a g_{bk}$$

$$= -g^{ik} \partial_k \ln \sqrt{-g} + g^{ab} g^{ik} \partial_a g_{bk}$$ By using (19)

$$= -g^{ik} \partial_k \ln \sqrt{-g} - \partial_a g^{ai}$$

$$= -\frac{g^{ik} \partial_k \sqrt{-g}}{\sqrt{-g}} - \partial_a g^{ai} \frac{\sqrt{-g}}{\sqrt{-g}}$$

Rem

$$g^{ab} \Gamma^i_{ab} = -\frac{1}{\sqrt{-g}} \partial_k \left(g^{ik} \sqrt{-g}\right)$$

3) ∴ we know 2 exp. of Γ in terms of metric tensors.

(21) & (22)

4) $\nabla_i A^i$ = Generalization of $\partial_i A^i$

$$(\nabla_i A^i) = \partial_i A^i + \Gamma^i_{ji} A^j$$

from (21)

$$= \partial_i A^i + A^j \frac{g^{ik} \partial_j g_{ki}}{2}$$

$$= \partial_i A^i + \frac{A^j}{\cancel{2}} \partial_j \ln(\sqrt{-g})$$

$$= \partial_i A^i + \frac{A^j \partial_j \sqrt{-g}}{\sqrt{-g}}$$

$$= \frac{\sqrt{-g}\,\partial_i A^i + A^i \partial_i \sqrt{-g}}{\sqrt{-g}}$$

~~Very~~

$$\boxed{\nabla_i A^i = \frac{\partial_i(\sqrt{-g}\,A^i)}{\sqrt{-g}}}$$

(25) Example in Spherical coord.

$$\nabla \cdot \vec{v} = \frac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\,v^i\right)$$

$$= \frac{1}{r^2\sin\theta}\,\partial_\alpha\left(r^2\sin\theta\, v^\alpha\right)$$

$$= \frac{1}{r^2}\,\partial_r\left(r^2 v^r\right) + \frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\, v^\theta\right) + \partial_\phi\left(v^\phi\right)$$

~~$\nabla = \frac{1}{r^2}\partial_r$~~

(26) $$\int d^4x\, \partial_i A^i \overset{SR}{=} \int_{\partial V} d^3x\, A^i n_i$$

$$\int_V \sqrt{-g}\, d^4x\, \nabla_i A^i = \int d^4x\, \partial_i(\sqrt{g}\,A^i)$$

This is no more diff. than putting $r^2\sin^2\theta\, dr\, d\theta\, d\phi$

Now (SR) trick can't be played as $\sqrt{-g}$ is there

∴ use (24)

$$= \int_{\partial V} \boxed{d^3x\, \sqrt{h}}\, (n_i A^i)$$

$\sqrt{h}$ → Det. of metric on Surface.

↳ Volume element of Surface

(27) $S = f(x^i) = 0$ (Surf...

(Normal) $\partial_i f = n_i$ in SR & GR.

Local things should match & ∴ it matches.

as U^i is the local thing & it should match

∴ as Normal is local thing the def. ∴ $n_i = \partial_i f$.

(28) Surface

3 Dimensional

$x^a = x^a(y^1, y^2, y^3)$

3D Surface should be described by 3 variables

(29) $ds^2 = g_{ab}\, dx^a\, dx^b$

$$ds^2 = \left(g_{ab} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta}\right) dy^\alpha dy^\beta$$

$dy^\alpha dy^\beta$ → 3D

$\left(g_{ab} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta}\right) = h_{\alpha\beta}$ → Metric Induced on Surface

∴ Given Surface & metric we can write down $h_{\alpha\beta}$

$$|h_{\alpha\beta}| = h$$

(30) $F_{ik} = \partial_i A_k - \partial_k A_i$ ⟷ Not Covariant ∴ we want generalizat...

$$F_{ik} = \nabla_i A_k - \nabla_k A_i = \partial_i A_k - \partial_k A_i$$

∴ Definition field Tensor Doesn't pick any Correction

(31) $\partial_i F^{ik} = 4\pi J^k$

↓ Generalization

$$\nabla_i F^{ik} = \partial_i F^{ik} + \Gamma^i_{mi} F^{mk} + \Gamma^i_{mi} F^{im}$$

Sym ↙ Antisym ↘

$$= \partial_i F^{ik} + \frac{g^{ik}\partial_m g_{ki}}{2} F^{mk}$$

$$= \partial_i F^{ik} + \partial_m (\ln \sqrt{-g}) F^{mk}$$

$$= \partial_i F^{ik} + \frac{\partial_m \sqrt{-g}}{\sqrt{-g}} F^{mk}$$

$$= \frac{\sqrt{-g}\,\partial_i F^{ik} + (\partial_i \sqrt{-g}) F^{ik}}{\sqrt{-g}}$$

$$\nabla_i F^{ik} = \frac{\partial_i (\sqrt{-g} F^{ik})}{\sqrt{-g}}$$

Compare with (24) $\nabla_i A^i$

(32) <u>Maxwell Eqn in Curved Spacetime</u>

$$\partial_i F^{ik} = 4\pi J^k \longrightarrow \nabla_i F^{ik} = \frac{\partial_i (\sqrt{-g} F^{ik})}{\sqrt{-g}} = 4\pi J^k$$

In the same way

$$u^i \partial_i u^a = 0 \longrightarrow u^i \nabla_i u^a = 0$$

(33) Bending of light
&
Effect of ~~EM on~~ GR on EM
are all included in this

(34)
$$\frac{\partial_i(\sqrt{-g}\,F^{ik})}{\sqrt{-g}} = 4\pi J^k$$

Raising & lowering of indices of F^{ik}
Now done by g_{ik}.

(35)
$$\partial_k \partial_i (\sqrt{-g}\,F^{ik}) = 4\pi\, \partial_k (\sqrt{-g}\,J^k)$$

$$= \frac{\partial_k \partial_i (\sqrt{-g}\,F^{ik})}{\sqrt{-g}} = \frac{4\pi\, \partial_k (\sqrt{-g}\,J^k)}{\sqrt{-g}}$$

Sym ($\partial_k \partial_i$), Ant. (F^{ik}) → 0

$$= 4\pi \nabla_k J^k$$

$\Rightarrow 4\pi \nabla_k J^k = 0$ (Charge Conservation in General Spacetime)

(36)
$$A_{EM} \overset{SR}{=} -\int d^4x\, A_k J^k - \frac{1}{16\pi}\int d^4x\, F_{ik}F^{ik}$$

$$A_{EM} \overset{GR}{=} -\int \sqrt{-g}\, d^4x\, A_k J^k - \frac{1}{16\pi}\int \sqrt{-g}\, d^4x\, F_{ik}F^{ik}$$

(37) **Postulate:**

All gravitational effects are geometrical

(38) **Scalar field**

Massless

$$\partial_i \partial^i \phi \overset{\text{SR}}{=} 0 = \Box \phi$$

~~AS~~

$$\nabla_i \nabla^i \phi = \frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g}\, g^{ik} \partial_k \phi \right) = 0$$

Laplacian

① $\nabla_a B^a = \frac{\partial_a(\sqrt{-g}\, B^a)}{\sqrt{-g}}$

$\int d^4x \sqrt{-g}\, \nabla_a B^a = \int_{\partial V} d^3x \sqrt{h}\, n_a B^a$

Taking time = const. surface & then $\int_{t=t_1}^{t=t_2} d^3x \sqrt{h}\, B^0$

② from ch-2 (39) (42)

$\partial_a T^a{}_b = 0$

$\frac{\text{in SR}}{0} = \int d^4x\, \partial_a T^{ab} = \int_{t=t_1}^{t=t_2} d^3x\, T^{0b} = 0$

↑ 4 momentum conserved

~~But then would Anti Symmetric tensor cause~~

But in GR this doesn't hold true.

③ As T^{0b} is Tensor with component b free we are adding ~~different~~ vectors when we are integrating

Imp.

But that doesn't holds closure property. ∴ It is not covariant.

④ In ① $n_a B^a$ was a scalar so we did not get into trouble

In ② $n_a T^{ab} \Rightarrow T^{0b}$

What about Ant. Sym Tensor?

⑤ Generalization of $\partial_a T^{ab} = 0$

$\nabla_a T^{ab} = 0$ → in scalar field.

⑥ $\nabla_a T^{a}{}_{b} = \partial_a T^{a}{}_{b} + \Gamma^{a}_{ma} T^{m}{}_{b} - \Gamma^{m}_{ab} T^{a}{}_{m}$

$$= \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, T^{a}{}_{b}\right) - T^{ma} \underbrace{\Gamma_{m,ba}}$$

$\frac{1}{2}(-\partial_m g_{ab} + \partial_a g_{bm} + \partial_b g_{am})$

$$\boxed{\nabla_a T^{a}{}_{b} = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, T^{a}{}_{b}\right) - \frac{1}{2}(\partial_b g_{am}) T^{am}}$$

⑦ $\int d^4x \, \nabla_a T^{ab}$

$$= \int d^4x \left(\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, T^{a}{}_{b}\right) - \frac{1}{2}(\partial_b g_{am}) T^{am} \right)$$

If g_{am} is ind. of b coordinate then as in ① we can integrate.

Ex. If g_{am} is ind. of time coordinate then $\int T^{a}{}_{0}\, d^3x$ = 4 momen. will be conserved.

⑧ $\xi^a(x)$; → vector $(T^{ab} \xi_b)$

If ξ^a has time component only then $T^{ab}\xi_b$ is just equivalent to $T^{a}{}_{0}$ which in ⑦ is conserved

⑨ $\nabla_a (T^{ab} \xi_b) = \nabla_a T^{ab} + (\nabla_a \xi_b) T^{ab}$

$$= T^{ab} (\nabla_a \xi_b)$$

If $\nabla_a \xi_b$ is antisymmetric then $\nabla_a (T^{ab} \xi_b) = 0$

12.

(10) $0 = \int d^4x \sqrt{-g}\, \nabla_a (T^{ab} \xi_{\varphi b}) = \int_{\partial V} d^3x \sqrt{h}\, T^{ob} \xi_{\varphi b}$

just as in (1)

covaraint as it is scalar as there is no hanging indêces.

(11) But conservation law holds only if $\xi_{\varphi a}$ holds condition & ξ exist

$\nabla_a \xi_{\varphi b} + \nabla_b \xi_{\varphi a} = 0$] Killing Equation

time like vector $= \xi_{\varphi a}$: Killing vector $\equiv (\xi_\varphi, 0, 0, 0)$

~~killing vector to ∃ in spacetime~~

~~(12) We assume~~

(12) There is no guarantee that a killing vector would ∃ in spacetime.

(13) But if it ∃ then it gives conservation law if it follows killing eqn

(14) But from (7) we know that if g_{am} is ind. of some coordinate b s.t. $\partial_b g_{am} = 0$ & ∴ leads to conservation law

i.e. ∃ some symmetry in the metric for $\partial_b g_{am} = 0$ & conservation law to be hold.

(15) ∴ $\xi_{\epsilon a}$ ∃ only if some symmetry in space-time is there.

(16) Let $g_{ab}(t, \vec{x}) = g_{ab}(\vec{x})$ ind. of time.

then

if $x^a \longrightarrow x^a + \xi_\epsilon^a$

& $\xi_\epsilon^a = (\varepsilon, 0, 0, 0)$

the g_{ab} would remain same due to transfⁿ

(17) if $g_{ab}(t, \vec{x}) = g_{ab}(\vec{x})$ then by (7) conservation law holds.

& by (16) ∃ $\xi^a = (\varepsilon, 0, 0, 0)$ s.t. $g_{ab}(\vec{x}, t) = g_{ab}(\vec{x})$

Now putting ξ^a in ~~kill~~ $\nabla_a \xi_{\epsilon b} + \nabla_b \xi_a$ & using $g_{ab}(t, \vec{x}) = g_{ab}(\vec{x})$

we get killing eqⁿ

$\nabla_a \xi_{\epsilon b} + \nabla_b \xi_{\epsilon a} = 0$ ← ~~So the are Equivalent~~

if $g_{ab}(t,x) = g_{ab}(x)$ then ⇒ ∃ ξ_ϵ & kill. eqⁿ is satisf.

(18) Let $\xi_\epsilon^a = (\varepsilon, 0, 0, 0)$

ξ_ϵ^a = infinitesimal to first order

or

we can put $\epsilon \xi_\epsilon^a$ & take $\lim \epsilon \to 0$

Now under coordinate transfⁿ

$x^a \longrightarrow x^a + \xi_\epsilon^a(x)$

how g_{ab} changes?

(19) $\delta g^{ab} = \nabla^a \xi_\epsilon^b + \nabla^b \xi_\epsilon^a$ (Proof will be given later)

$g^{ab} \longrightarrow g^{ab} + \delta g^{ab}$

(20) Now if Killing eqn is valid then $\delta g^{ab} = 0$

∴ g^{ab} doesn't change ∴ Under some coord. tran. it will remain invari. ∴ Conservation law

(21) But why did we not do it in way (16) it wouldn't
Coz. in coordinates we would be using it wouldn't be clear which sym. is there.

(22) ∴ we want to characterise the symmetry of the spacetime in terms of independent Killing vector which ∃ in spacetime.

(23) Eg. if Kepler Problem is done in (x, y, z) coordi. we wouldn't be able to extract sym. But if done in (r, θ, ϕ) then we immediately see Angular mom. cons.

∴ Coordinates we are using hides symmetries.

∴ We need independent way of thinking about the Prob.

∴ we can characterise sym. of spacetime in terms of ind. Killing vector.

(24) $x^a \longrightarrow x^a + \xi^a(x) = x'^a$

$$g^{a'b'}(x') = \frac{\partial x^{a'}}{\partial x^i} \frac{\partial x^{b'}}{\partial x^j} g^{ij}(x)$$

$x' \equiv x$ in new coord. (Same Physical event in 2 coordinate)

(25) $g^{a'b'} = (\delta^a_i + \partial_i \xi^a_\varphi)(\delta^b_j + \partial_j \xi^b_\varphi)\, g^{ij}$

$g^{a'b'}(x') = g^{ab}(x) + g^{ib}\partial_i \xi^a_\varphi + g^{aj}\partial_j \xi^b_\varphi + O(\xi^2_\varphi)$.

$g^{a'b'}(x+\xi_\varphi) = g^{a'b'}(x) + \xi^k_\varphi \partial_k g^{a'b'}$ (Taylor Expn)

We want to know how the functional form of g_{ab} changes under our coordinate transformation. $g^{ik}(x^l)$ needs to be compared to $g^{ik}(x^l)$
i.e. $\delta g^{ab} = g^{ab}(x) - g^{a'b'}(x)$

$g^{a'b'}(x') - g^{ab}(x)$ = change in the components at given location.

$g^{a'b'}(x+\xi_\varphi) = g^{a'b'}(x) + \xi^k_\varphi \partial_k g^{a'b'}$

$f(x) = f(a) + f'(a)(x-a)$

As $g^{a'b'}(x) - g^{ab}(x) \sim \xi_\varphi$ (Diff. by infinitesimal order) why?

$\therefore g^{a'b'}(x+\xi_\varphi) = g^{a'b'}(x) + \xi^k \partial_k g^{ab}$.

(26) $\delta g^{ab} = g^{ib}\partial_i \xi^a_\varphi + g^{ia}\partial_i \xi^b_\varphi - \xi^k_\varphi \partial_k g^{ab}$

$= \nabla^a \xi^b_\varphi + \nabla^b \xi^a_\varphi$

by $\nabla^a \xi^b_\varphi = g^{ak}\nabla_k \xi^b_\varphi = g^{ak}(\partial_k \xi^b_\varphi + \Gamma^b_{ik}\xi^i_\varphi)$

$g^{ak} g^{bc}\Gamma_{cik}\xi^i_\varphi = \frac{g^{ak}g^{bc}}{2}\xi^i_\varphi(-\partial_c g_{ik} + \partial_i g_{ck} + \partial_k g_{ca})$

$\nabla^a \xi^b_\varphi = g^{ak}\nabla_k \xi^b_\varphi = g^{ak}(\partial_k \xi^b_\varphi + \Gamma^b_{mk}\xi^m_\varphi)$

$g^{ak}\Gamma^b_{mk}\xi^m_\varphi = g^{ak}g^{bc}\Gamma_{cmk}\xi^m_\varphi = \frac{g^{ak}g^{bc}\xi^m_\varphi}{2}(-\partial_c g_{mk} + \partial_m g_{kc} + \partial_k g_{mc})$

$\nabla^b \xi^a_\varphi = g^{bk}\nabla_k \xi^a_\varphi$

$g^{bk}\Gamma^a_{mk}\xi^m_\varphi = \frac{g^{bk}g^{ac}\xi^m_\varphi}{2}(-\partial_c g_{mk} + \partial_m g_{kc} + \partial_k g_{mc})$

(27) $\delta g_{ib} = -(\nabla_a \xi_b + \nabla_b \xi_a)$ where $g_{i'k'} = g_{ik} + \delta g_{ik}$

(28) δg_{ib} is not a Tensor — Doubt
so don't get δg^{ib} from δg_{ib} by Raising & lowering Indices

Inverse ? ?

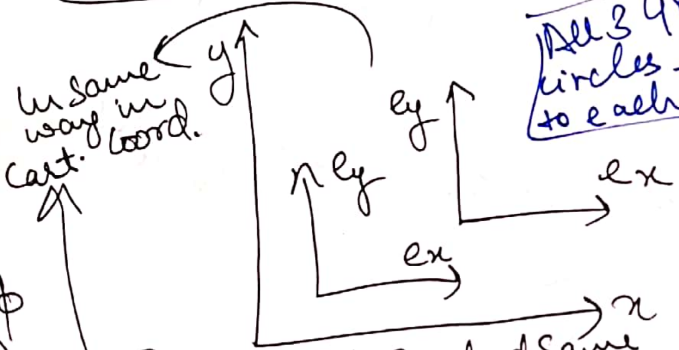
(29) Think of $\delta(g_{ib} g^{bc})$ — 0

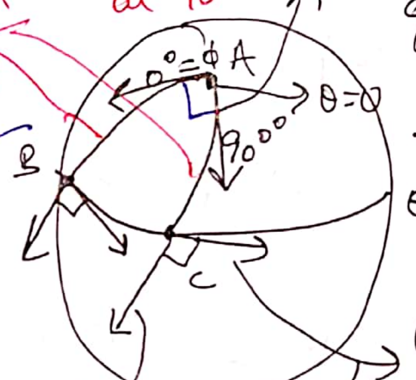
Ans $(\delta g^{ab}) g_{bc} g_{ad} = -g^{ab} \delta g_{bc} g_{ad}$

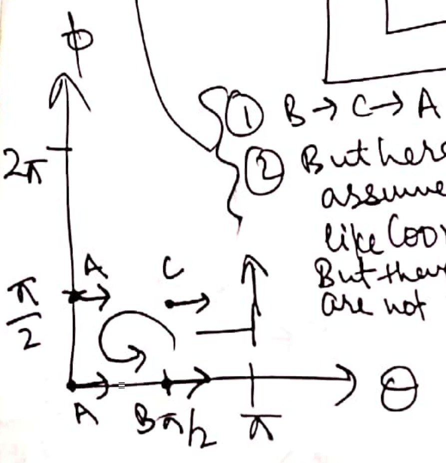
$\Rightarrow (\delta g^{ab}) g_{bc} g_{ad}$

By Extremizing action we get geodesic Eqn. But the Soln of these geodesics may not be Unique

(30) Parallel transport







∴ Going from A → B → C → A
Doing // transport vector has changed.
This is the intuitive way of understanding curvature

(31) But in Cartesian coordinate
// transport vector ~~has~~ will come back to itself.

Scanned with CamScanner

Killing Vector

(1) Any vector that satisfies Killing Eqn $\nabla_a \xi_b + \nabla_b \xi_a = 0$ is called a Killing vector.

(2) Killing Eqn provides an operational way of determining the symmetries of metric Tensor.

(3) By solving Killing Eqn we can determine all indep. vector fields which satisfy this eqn. The integral curves to these vector fields will define the directions of the symmetries of spacetime. Along these integral curves, metric will remain invariant.

(4) The symmetry implies $\exists$ Killing vector ξ^a and satisfies Killing Eqn

(5) $u^i \xi_i$ is conserved along the Geodesic

$$\frac{d(u^i \xi_i)}{d\lambda} = u^b \nabla_b (u^i \xi_i) = \xi_i u^b \nabla_b u^i + u^i u^b \nabla_b \xi_i = 0$$

) Similarly $T^{ab} \xi_b \equiv P^a$ is conserved.

$$\nabla_a (T^{ab} \xi_b) \equiv \nabla_a P^a = 0$$

) $\nabla_a T^a{}_b = 0$ identity.

) $\nabla_a (T^{ab} \xi_b) = \xi_b \nabla_a T^{ab} + T^{ab} \nabla_a \xi_b$

$$= T^{ab} \nabla_a \xi_b$$

But if $\nabla_a \xi_b + \nabla_b \xi_a = 0$

then $\nabla_a (T^{ab} \xi_b) = 0$

(32) A vector is constant when we define a procedure to move a vector // to itsef from one pt. to another.

In Cartesian Coordinate

(33) constant vector: (1) Vectors whose compoments are constant w.r.t cartesian cordinates

(2) Vector field can be thought of // to itself at every pt.

(34) Now in general, to define a constant vector, we need a well defined procedure to move a vector // to itself.

(35) In Cartesian coordinates, by keeping the components const. we can move vector // to itself. ∴ We have found a procedure ∴ constant is defined.

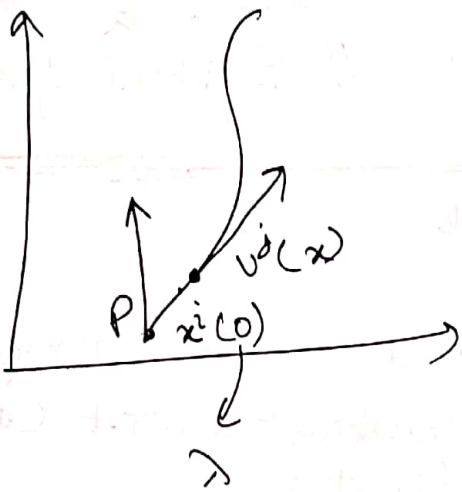
→ This def. matches with (33) (1) in flat spacetime

(36) In flat spacetime in curvilinear coordinate, we can introduce cartesian coordinate globally & then def. constant vector there & then again transform back to curv. coord. ∴ In flat spacetime we can always // transp. irr. of coord system

(37) But in a curved spacetime it is not possible to transf. to cart. coord. globally. Therefore (36) fails

 Generalization of

(38)



$\left(\frac{dx^b}{d\lambda}\right)\nabla_b v^a \rightarrow$ Directional derivative of the vector field along a particular direction.

Any vector const vector can be // transport

Claim: This is the def. of // transport

(39) $x^i(\lambda) \longrightarrow \frac{dx^i}{d\lambda} = v^i$

let $v^i \nabla_i k^j = 0 \longrightarrow$ To be solved for k^j

x^i is given $\therefore v^i$ is given

we know metric.

Directional Derivative of vector vanishes along the tangent vector to given curve.

(40) As $v^i \nabla_i k^j = 0$ is the def. of const. vector in cart. coord. we can

And for const. vector, transport $\|^{ly}$.

$\therefore$ It is reasonable that $v^i \nabla_i k^j = 0$ is the def. of Parallel transport.

in Cartesian coordinate $\Gamma = 0 \therefore \frac{dk^j}{d\lambda} = 0$

i.e. Components are same $\Rightarrow$ // transport,

(41) $\frac{dx^i}{d\lambda}\left\{\partial_i k^j + \Gamma^j_{li} k^l\right\} = 0$

This is homogen 1st order DE.

$\frac{dk^j}{d\lambda} + \Gamma^j_{li} k^l \left(\frac{dx^i}{d\lambda}\right) = 0 \rightarrow$ we know $\Gamma^i_{jl}(\lambda)$ $x^i(\lambda)$

we will get to know k^j in space $\therefore$

(42) We know metric everywhere in space.

We know metric along the curve.

We know metric along the curve

$\therefore$ We know $\Gamma^j_{li}(x)$ along

And as $x^i(\lambda)$ $\Gamma^j_i(\lambda)$

(43) So from (41)

we will get $k^j(\lambda)$.

But it will req. 1 Initial condition, as it is 1st order DE.

i.e. let $k^j(0)$ is given

(44) ∴ at any λ we can find k^j which is // transport.

(45) from geodesic Eqn → ∴ Tangent vector // transport itself

$v^i \nabla_i v^a = 0$

where v^a is tangent to curve $\frac{dx^a}{d\lambda} = v^a$.

But in (40) → let $k^j = v^j$

$v^i \nabla_i k^j = 0$

k^j is any vector at that curve.

(46) ∴ from (40) ~~we find any vector which is // to itself along curve~~

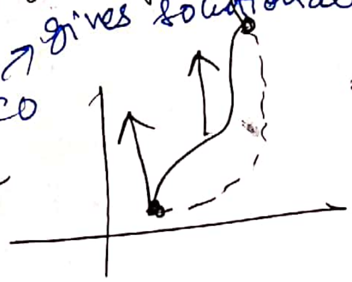
& ~~also this (40) tells that we can find~~ tangent ~~any~~ vector which is // to ~~itself along curve~~ or

Tangent vector ~~// transport itself along curve~~

(47) This we already know from Geodesic Eqn.

(48) ∴ Given any curve I can // transport any vector along the curve. ↙ assuming $v^i \nabla_i k^j = 0$ ↗ gives solutions always

(49) Now let us take another curve keeping the end pts fixed



∴ We took another $x^{i'}(\lambda)$

& ∴ Γ^{j}_{ei} changes

Will we get same $k^{j}(\lambda)$ at the end pt?

in Sphere we didn't get the same $k^{j}(\lambda)$.

∴ in General we will not get same $k^{j}(\lambda)$

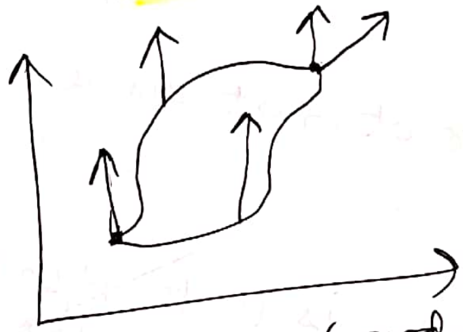
(50) Surely, for the other curve

We will end up getting ∥ transport $k^{j}(\lambda)$

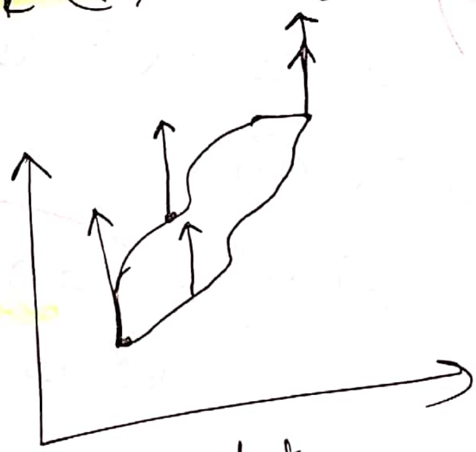
But will $k^{j}(\lambda)$ be same at end pt?

(87) Th. But in Cartesian Coordinate

I am guaranteed that $k^{j}(\lambda)$ along any curve will be same.



in Curv. ~~Coord~~ Spacetime



in flat Spacetime

Proof: $v^{i}\nabla_{i}k^{j}=0$

in flat spacetime, go to cart. coord. globally.

∴ $\Gamma^{j}_{ei}=0$

∴ $\frac{dk^{j}}{d\lambda}=0$ ∴ k^{j} is the unique soln

(52) In Curved spacetime,

$$\frac{dK^j}{d\lambda} + \Gamma^j_{li} K^l \left(\frac{dx^i}{d\lambda}\right) = 0 \quad \text{— (1)}$$

does not in general gives me the Unique sol^n.

∴ Under what cond^n will it give me the Unique sol^n?

(53) from (41)

$$\frac{dx^i}{d\lambda} \left\{ \partial_i K^j + \Gamma^j_{li} K^l \right\} = 0 \quad \text{— (2)}$$

Condition for (1) to give Unique sol^n is $\partial_i K^j + \Gamma^j_{li} K^l = 0$

Bec in (2) $\frac{dx^i}{d\lambda}$ is the part depending on curve

other part is ind. of curve

∴ if that is 0 then that will give me the parallel transport vector which is ind. of any curve.

(54) $\partial_i K^j = -\Gamma^j_{li} K^l \rightarrow$ Partial DE.

These PDE has Integrability cond^n Under which sol^n will ∃ otherwise not.

(55) Integribility cond.

$\partial_m \partial_i K^j = -\partial_m \partial_i (\Gamma^j_{li} K^l)$ — (1)

By this we get cond^n on Γ

if this cond^n is satisfied we has sol^n to PDE

(1) $dK^j + \Gamma^j_{li} K^l dx^i = 0$

$dK^j = -\Gamma^j_{li} K^l dx^i$

K^j as f^n of x^i

if RHS is given there is no guarante LHS dK^j. Under some cond^n it will hold.

(56) $\partial_i k^j = -\Gamma^j_{\ell i} k^\ell$

$\partial_m \partial_i k^j = -(\partial_m \Gamma^j_{i\ell}) k^\ell - \Gamma^j_{i\ell} \partial_m k^\ell$

$= -(\partial_m \Gamma^j_{i\ell}) k^\ell + \Gamma^j_{i\ell} \Gamma^\ell_{pm} k^p$

This is linear in k $\therefore$ if k_1 & k_2 are sol^n then $c_1 k_1 + c_2 k_2$ is the sol^n also

(57) $\partial_m \partial_i k^j = \left(-(\partial_m \Gamma^j_{ip}) + \Gamma^j_{i\ell} \Gamma^\ell_{pm}\right) k^p$

$0 = (\partial_m \partial_i - \partial_i \partial_m) k^j = \left(-\partial_m \Gamma^j_{ip} + \Gamma^j_{i\ell} \Gamma^\ell_{pm} + \partial_i \Gamma^j_{mp} - \Gamma^j_{m\ell} \Gamma^\ell_{pi}\right) k^p$

Assuming $\partial_i k^j + \Gamma^j_{\ell i} k^\ell = 0$ has a sol^n & we can Rep. $\partial_i k^j = -\Gamma^j_{\ell i} k^\ell$

$= -\left(\partial_m \Gamma^j_{ip} - \partial_i \Gamma^j_{mp} - \Gamma^j_{i\ell} \Gamma^\ell_{pm} + \Gamma^j_{m\ell} \Gamma^\ell_{pi}\right) k^p$

$= -R^j{}_{pmi} k^p$

Riemann Christofell Tensor or Curvature Tensor

$j, p \equiv$ Matrix indices
$m, i \equiv$ Spacetime indices

(58) $R^j{}_{pmi} k^p \equiv k \left(\partial_m \Gamma_i - \partial_i \Gamma_m + [\Gamma_m, \Gamma_i]\right)$

$[\Gamma_m, \Gamma_i]$ ← commutator

Yangmills Theory

In Gauge Field theory

$\Gamma \equiv$ Connections

$i \equiv$ spacetime index

$j, p \equiv$ internal Gp indices.

(59) The condition that ∃ vector field in spacetime when parallel transported in unique way i.e. ind. of curve on which it is ∥ transported $\equiv R^{j}_{pmi} = 0$

(60) ∴ in Flat spacetime (in any coord. system)

$R^{j}_{pmi} K^{p} = 0$

$\Rightarrow R^{j}_{pmi} = 0$

∴ R if vanishes in one coord. system then it vanishes in any other coord. system

∴ It should be a Tensor

(61) R is 2^{nd} derivative of metric

Metric ≡ Potential

Γ ≡ Forces

R ≡ derivative of forces.

R → This quantifies Curvature

(62)

(63) if R = 0 & I have been given basis at P then I can Parallel transport them anywhere & have global Basis.