Scalar field dynamics in black hole backgrounds

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Overview

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Academic Activities

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Motivation

- **Stability of black holes**: Given the initial perturbation, will the perturbation remain bounded at all times?

  [Regge, Tullio and Wheeler, John A. (*Phys. Rev* 1957)]

- **Gravitational wave Astronomy**: To determine different chacterstics of the black hole like their mass, charge, and angular momentum.
- **Confirmation of BHs**: Detection of these frequencies also confirm the existence of the BHs.

The aim of the project

What will happen to the black hole when it is perturbed by an external field?

- A perturbed black hole will go through quasinormal modes ringdown. The aim of the project was to calculate these quasinormal modes.

Presentation Plan

- Klein Gordan Equation in Schwarzschild metric and Reissner–Nordström metric.
- Calculating black hole quasinormal modes by using WKB method (approximation)
- Calculating black hole quasinormal modes by using Leaver’s method (exact numerical solution)

Scalar Perturbation-Introduction

- Klein Gordan Equation for massless scalar field in spherically symmetric and static background

$$\Box\phi = 0$$

- Decomposing the function as

$$\phi(t, r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\Psi_{\ell m}^{s=0}(r)}{r} P_{em}(\theta) e^{-i\omega t} e^{im\phi}$$

- By using tortoise coordinate r_* defined as $\frac{dr}{dr_*} \equiv (fh)^{1/2}$, we get the Schrodinger equation:

$$\frac{d^2\Psi_l^{s=0}}{dr_*^2} + \left[\omega^2 - V_0\right] \Psi_l^{s=0} = 0$$

where effective potential reads as

$$V_0(\mu) \equiv f\frac{\ell(\ell+1)}{r^2} + \frac{(fh)'}{2r}$$

Schwarzschild metric

- Tortoise coordinate

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$r_\star \to -\infty$ (corresponds to $r \to 2M$)
$r_\star \to \infty$ (corresponds to $r \to \infty$)

- Effective Potential

$$V_0^{\text{Schw}}(\mu) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)$$

- $V_0^{\text{Schw}} \to 0$ asymptotically.
- In both regions, wave equation behave as

$$\phi \sim e^{-i\omega(t \pm r_*)}$$

Boundary Conditions

- **Horizon**: Classically, nothing should leave horizon, therefore

$$\phi \sim e^{-i\omega(t+r_+)}$$

- **Infinity**: Discarding unphysical waves entering from infinity.

$$\phi \sim e^{-i\omega(t-r_+)}$$

- Now our aim is to solve for ω in the radial equation satisfying the above boundary conditions.
- I will show how these boundary conditions make eigen functions damped in nature.

Master's Equation

- For any massless spin (s=0 scalar, s=1 Electromagnetisc, s=2 Gravitational):

$$\frac{d^2\Psi_\ell^s}{dr_*^2} + \left[\omega^2 - V_s\right]\Psi_\ell^s = 0$$

where

$$V_s = f\left(\frac{\Lambda}{r^2} + \frac{2\beta}{r^3}\right)$$

$\beta = 1 - s^2$
$\lambda = \ell(\ell + 1)$

- $$\frac{d^2\Psi}{dx^2} + Q(x)\Psi = 0$$

WKB Method

- If $E < V(x)$, then the reflected amplitude is generally comparable to the incident amplitude, while the transmitted amplitude is much smaller.
- In case of black holes, the transmitted and reflected waves have comparable amplitudes.
- **Validity**: Turning points degenerate or very close to each other.

WKB Method

- In region II, expanding Q about maxima,

$$Q(x) = Q_0 + \frac{1}{2} Q_0'' (x - x_0)^2 + \mathcal{O}(x - x_0)^3$$

- Let $k \equiv Q_0''/2 \quad t = (4k)^{1/4} e^{\mathrm{i}\pi/4} (x - x_0) \quad \nu + \frac{1}{2} = -\frac{\mathrm{i} Q_0}{(2Q_0'')^{1/2}}$
- We get the final equation whose soultions are parabolic cylinder functions,

$$\frac{d^2\Psi}{dt^2} + \left[\nu + \frac{1}{2} - \frac{1}{4} t^2\right] \Psi = 0$$

- By applying BC, we get

$$n + \frac{1}{2} = -\frac{\mathrm{i} Q_0}{(2Q_0'')^{1/2}}$$

- Finally, we get

$$\omega_n^2 = i \left(n + \frac{1}{2}\right) \sqrt{2Q_0''} + \left(1 - \frac{2M}{r_0}\right) \left(\frac{\Lambda}{r_0^2} + \frac{2\beta M}{r_0^3}\right)$$

WKB Method

- In eikonal limit, we can relate QNM to lyapunov exponent

$$\omega_{QNM} = \Omega_c l + i\left(n + \frac{1}{2}\right)|\lambda|$$

- When taking approximation to 6th order, we get

$$\omega^2 = \left[V_0 + \left(-2V_0''\right)^{1/2}\tilde{\Lambda}(n)\right] - \mathrm{i}\alpha\left(-2V_0''\right)^{1/2}\left[1 + \tilde{\Omega}(n)\right]$$

where

$$\tilde{\Lambda}(n) = \frac{1}{\left(-2V_0''\right)^{1/2}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4} + \alpha^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2\left(7 + 60\alpha^2\right)\right]$$

$$\tilde{\Omega}(n) = \frac{1}{\left(-2V_0''\right)}\left[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4\left(77 + 188\alpha^2\right) - \frac{1}{384}\frac{V_0'''V_0^{(4)}}{V_0''^3}\left(51 + 100\alpha^2\right)\right.$$
$$+ \frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2\left(67 + 68\alpha^2\right) + \frac{1}{288}\frac{V_0'''V_0^{(5)}}{V_0''2}\left(19 + 28\alpha^2\right)$$
$$\left. - \frac{1}{288}\frac{V_0^{(6)}}{V_0''}\left(5 + 4\alpha^2\right)\right]$$

Application - Schwarzschild metric

For scalar field, s=0

l	n	σ_{WKB}
0	0	$0.104648 - 0.115196i$
	1	$0.0891898 - 0.354959i$
	2	$0.063479 - 0.594572i$
	3	$0.0255008 - 0.83504i$
1	0	$0.291114 - 0.0980014i$
	1	$0.262212 - 0.307432i$
	2	$0.223543 - 0.52681i$
	3	$0.173702 - 0.748629i$
2	0	$0.483211 - 0.0968049i$
	1	$0.463192 - 0.29581i$
	2	$0.43166 - 0.503433i$
	3	$0.392578 - 0.715869i$

For Electromagnetic field, s=1

l	n	σ_{WKB}
1	0	$0.291114 - 0.0980014i$
	1	$0.262212 - 0.307432i$
	2	$0.223543 - 0.526817i$
	3	$0.173702 - 0.748629i$
2	0	$0.483211 - 0.0968049i$
	1	$0.463192 - 0.29581i$
	2	$0.43166 - 0.503433i$
	3	$0.392578 - 0.715869i$
3	0	$0.675206 - 0.096512i$
	1	$0.660414 - 0.2923441i$
	2	$0.634839 - 0.494118i$
	3	$0.602182 - 0.701053i$

Application - Schwarzschild metric

For gravitational field, s=2

l	n	σ_{WKB}
2	0	$0.373162 - 0.0892174i$
	1	$0.346017 - 0.274915i$
	2	$0.302935 - 0.471064i$
	3	$0.247462 - 0.672898i$
3	0	$0.599265 - 0.0927284i$
	1	$0.582355 - 0.281406i$
	2	$0.5532 - 0.476684i$
	3	$0.515747 - 0.677429i$
4	0	$0.809098 - 0.0941711i$
	1	$0.796499 - 0.284366i$
	2	$0.773636 - 0.478974i$
	3	$0.743312 - 0.6783i$

Leaver's Method

- Standard Radial Equation,

$$r(r-1)\frac{d^2\Psi_l^s}{dr^2}+\frac{d\Psi_l^s}{dr}-\left[\ell(\ell+1)-\frac{s^2-1}{r}-\frac{\omega^2 r^3}{r-1}\right]\Psi_l^s=0$$

- Using the series Expansion, $\Psi_l^s=(r-1)^{-\mathrm{i}\omega}r^{2\mathrm{i}\omega}e^{\mathrm{i}\omega(r-1)}\sum_{j=0}^{\infty}a_j\left(\frac{r-1}{r}\right)^j$ satisfying boundary conditions
- We get three term recursion relation, $\alpha_0 a_1+\beta_0 a_0=0$
  $\alpha_n a_{n+1}+\beta_n a_n+\gamma_n a_{n-1}=0, \quad n=1,2,\ldots$
- Applying the condition that series should be convergent

$$0=\beta_0-\frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\ldots$$

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Thank You