Action Formulation in General Relativity

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Overview

Basics

The discussion throughout the talk will be on arbitrary spacetime (M,g) with connections defined on it.

- For affinely parameterized curve, geodesic equation is : $U^{\alpha}\nabla_{\alpha}U^{\beta} = 0$
- For any two vector fields intersecting (as shown below) the lie derivative of one vector field w.r.t other vanishes.

$$\mathcal{L}_u\xi^{\alpha} = \mathcal{L}_{\xi}u^{\alpha} = 0 \quad \Rightarrow \quad \xi^{\alpha}_{;\beta}u^{\beta} = u^{\alpha}_{;\beta}\xi^{\beta}$$

For affinely parameterized geodesic : $u^{\alpha}\xi_{\alpha} = 0$

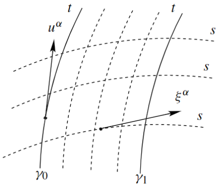

Figure: Deviation vector between two neighbouring geodesics

Basics of Timelike Congruence

A congruence is a family of curves such that through each point there passes only one curve from this family.

- For affine geodesics, the following relationships holds:
  $u^{\alpha}u_{\alpha} = 1, \quad u^{\alpha}_{;\beta}u^{\beta} = 0, \quad u^{\alpha}_{;\beta}\xi^{\beta} = \xi^{\alpha}_{;\beta}u^{\beta}, \quad u^{\alpha}\xi_{\alpha} = 0$
  Deviation vector points in the directions transverse to the flow of the congruence.
- Spacetime metric $g_{\alpha\beta}$ can be decomposed into longitudinal and transverse part as $h_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha}u_{\beta}$
- Evolution of Deviation vector: $\xi^{\alpha}_{;\beta}u^{\beta} = B^{\alpha}_{\beta}\xi^{\beta}$
- $B_{\alpha\beta}$ is purely transverse : $B_{\alpha\beta}u^{\beta} = u^{\alpha}B_{\alpha\beta} = 0$
  Hence we can decompose it as : $B_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{3} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$

Expansion Parameter : $\theta = \nabla_i u^i$
This parameter describes the fractional rate of change of congruence's volume.
Shear Tensor $\sigma_{\alpha\beta}$: This is a symmetric and traceless quantity, which describes how the shape of the congruence changes.
Rotation Tensor $\omega_{\alpha\beta}$: This is the antisymmetric part of B_{ij} which describes how the congruence of geodesics will rotate.

Theorems for Timelike Geodesics

- **Frobenius Theorem**: Congruence of Curves is Hypersurface Orthogonal iff
$$\nabla_{[i}U_j U_{k]} = 0$$
For timelike geodesic, rotation parameter should vanish $\omega_{ab} = 0$
- **Raychaudhuri's Equation**: The evolution equation for the expansion scalar:
$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{ij}u^i u^j$$
- **Focusing Theorem**: For matter following Strong energy condition $R_{\alpha\beta}u^\alpha u^\beta \geq 0$ and the geodesic congruence be hypersurface orthogonal, the expansion must decrease during its' evolution.
$$\frac{d\theta}{d\tau} \leq 0$$
For $\theta = \theta_0 < 0$ under these conditions θ goes to $-\infty$ along the geodesic within the proper time $\tau \leq \frac{3}{|\theta_0|}$
The congruence will develop a caustic within finite proper time.

Basics of Null Geodesics

- For affine geodesics, the following relationships holds:
  $k^\alpha k_\alpha = 0, \quad k^\alpha_{;\beta} k^\beta = 0, \quad k^\alpha_{;\beta}\xi^\beta = \xi^\alpha_{;\beta} k^\beta, \quad k^\alpha \xi_\alpha = 0$
  Last condition fails to remove component of ξ^α in the direction of k^α and hence, deviation vector doesn't points in the directions transverse to the flow of the congruence.
- The transverse metric is 2-Dimensional, hence to isolate transverse part of the metric we need two null vectors. Spacetime metric $g_{\alpha\beta}$ can be decomposed as

  $$g_{\alpha\beta} = h_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

  where $N^\alpha N_\alpha = 0$ and $k^\alpha N_\alpha = 1$ do not determine N^α uniquely.
- Evolution of Deviation vector: $\xi^\alpha_{;\beta} k^\beta = B^\alpha_\beta \xi^\beta$
- $B_{\alpha\beta}$ is not purely transverse : $B_{\alpha\beta} k^\beta = k^\alpha B_{\alpha\beta} = 0$ but $B_{\alpha\beta}$ is not orthogonal to N^α.
- By isolating transverse component we get:

  $$\tilde{\xi}^\alpha \equiv h^\alpha{}_\mu \xi^\mu = \xi^\alpha + (N_\mu \xi^\mu)\, k^\alpha$$
  $$\left(\bar{\xi}^\alpha_{;\beta} k^\beta\right)^\sim = \bar{B}^\alpha_\beta \bar{\xi}^\beta$$

  where $\bar{B}^\alpha_\beta = h^\alpha_\mu h^\nu_\beta B^\mu_\nu$

- $\bar{B}^{\alpha}_{\beta}\bar{\xi}^{\beta}$ can be interpreted as the transverse relative velocity between the neighbouring geodesics. Therefore, decomposing it as we did earlier:

$$\tilde{B}_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{2} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

Expansion Parameter : $\theta = \nabla_i k^i$
This parameter describes the fractional rate of change of null-congruence's cross sectional area.

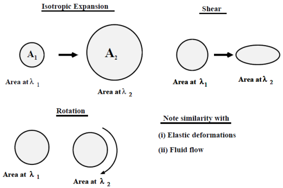

Figure: Physical interpretation of Parameters

Theorems for Null Geodesics

- **Frobenius Theorem**: Congruence of Curves is Hypersurface Orthogonal iff
$$\nabla_{[i} U_j U_{k]} = 0$$
For Null geodesics, rotation parameter should vanish $\omega_{ab} = 0$
- **Raychaudhuri's Equation**: The evolution equation for the expansion scalar:
$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{2} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{ij}k^i k^j$$
using $\tilde{B_{\alpha\beta}} \tilde{B^{\alpha\beta}} = B_{\alpha\beta} B_{\alpha\beta}$
$\boxed{\text{Weak Null Energy Condition} \iff \text{Strong Energy Condition}}$
- **Focusing Theorem**: For matter following Strong /Weak Null energy condition $R_{\alpha\beta}k^\alpha k^\beta \geq 0$ and the geodesic congruence be hypersurface orthogonal, the expansion must decrease during its' evolution.
$$\frac{d\theta}{d\lambda} \leq 0$$
For $\theta = \theta_0 < 0$ under these conditions θ goes to $-\infty$ along the geodesic within the affine parameter $\lambda \leq \frac{2}{|\theta_0|}$
The congruence will develop a caustic within finite proper time.

Theorems: Continued

- For Null Hypersurface, Null congruences K^{α} is tangent to the hypersurface, and thus are the null generators of the hypersurface.
- These null congruences satisfies the geodesic equation

$$k^{\alpha}\nabla_{\alpha}k^{\beta} = kk^{\beta}$$

Therefore, null hypersurface is generated by null geodesics.

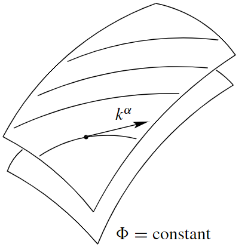

Figure: Family of Hypersurfaces orthogonal to congruence of null geodesics

Basics

In a four-dimensional spacetime manifold, a hypersurface is a three-dimensional submanifold that can be either timelike, spacelike, or null.

Timelike Hypersurface: Normal can be normalized $n_\alpha = \frac{\varepsilon \Phi,\alpha}{|g^{\mu\nu}\Phi,\mu \Phi,v|^{1/2}}$
where ϵ is 1 if hypersurface is spacelike and -1 if hypersurface is timelike.

Induced Metric or First fundamental form: Induced metric is 3 dimensional $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$

Normal n_α is defined such that $n_\alpha e^\alpha_a = 0$

Null Hypersyrface : Unit normal is not defined. We let $k_\alpha = \Phi_{,\alpha}$ making the hypersurface generated by the null geodesics.

As K^α is tangent to hypersurface, we can take null geodesics parameter as one of the coordinates of hypersurface $y^a = (\lambda, \theta^A)$

Induced Metric or First fundamental form: Induced metric is 2 dimensional as h becomes degenerate

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B, \quad e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda$$

Normal N_α is defined such that $N_\alpha e^\alpha_A = 0$, $N_\alpha K^\alpha = 1$

Integration on Hypersurfaces

- **Non-Null case**: Invariant three dimensional volume (surface element) on the hypersurface is $\mathrm{d}\Sigma \equiv |h|^{1/2}\, \mathrm{d}^3 y$
  $\mathrm{d}\Sigma_\mu = n_\alpha \mathrm{d}\Sigma$ is the directed surface element that points in the direction of increasing ϕ
- **Null case**: As h_{ab} is degenerate $h = 0$ and n_α does not exist. Therefore, the above expression should be generalized.
  Directed Surface element
  $$\mathrm{d}\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma} e_1^\alpha e_2^\beta e_3^\gamma \mathrm{d}^3 y$$
  This expression holds for both null and non-null hypersurfaces.
- Taking intrinsic coordinate y^1 as λ we can show $\mathrm{d}\Sigma_\mu = k^\nu\, \mathrm{d}S_{\mu\nu}\, \mathrm{d}\lambda$ where 2-dimensional surface element $\mathrm{d}S_{\mu\nu} = \varepsilon_{\mu\nu\beta_\gamma} e_2^\beta e_3^\gamma \mathrm{d}^2\theta$
  Further evaluating, we get
  $$\mathrm{d}S_{\alpha\beta} = 2k_{[\alpha} N_{\beta]} \sqrt{\sigma} \mathrm{d}^2\theta$$
- **Gauss Theorem**
  $$\int_V A^\alpha_{;\alpha} \sqrt{-g}\, \mathrm{d}^4 x = \oint_{\partial V} A^\alpha \mathrm{d}\Sigma_\alpha$$
  We can obtain conservation of charge if the divergence of A^α vanishes.

Tangent Vector Fields

- Tangent vector fields A^{α} purely tangent to hypersurface admit the decomposition
$$A^{\alpha} := A^{a} e^{\alpha}_{a}$$
- **Instrinsic Covariant Derivative**: Defined as the component of the projected $A_{\alpha;\beta}$, we get
$$\nabla_a A_b = \nabla_{\alpha} A_{\beta} e^{\alpha}_{a} e^{\beta}_{b} = A_{a,b} - \Gamma_{cab} A^{c}$$
where we have defined $\Gamma_{cab} = e^{\gamma}_{c} e_{a\gamma;\beta} e^{\beta}_{b}$
- Assuming this connection to be metric compatible and this connection to be symmetric in its last two argument we get the expression for connection:
$$\Gamma_{cab} = \frac{1}{2}\left(h_{ca,b} + h_{cb,a} - h_{ab,c}\right)$$

Extrinsic Curvature

- **Extrinsic Curvature or Second fundamental form** 1. Defined as the normal components of $\nabla_\beta A^\alpha e_b^\beta$ or 2. Defined as the component of the tagential covariant derivative of $\nabla_\beta n^\alpha$ we find,

$$K_{ab} = n_{\alpha;\beta} e_a^\alpha e_b^\beta$$

  We can get $k_{\alpha\beta}$ from the above expression: $k_{\alpha\beta} = \nabla_\alpha n_\beta - \epsilon n_\alpha n^\gamma \nabla_\gamma n_\beta$

- K_{ab} is a symmetric tensor. Therefore,

$$K_{ab} = n_{(\alpha;\beta)} e_a^\alpha e_b^\beta = \frac{1}{2}\left(\mathcal{L}_n g_{\alpha\beta}\right) e_a^\alpha e_b^\beta$$

Physical Interpretation : K_{ab} is related to normal derivative of the metric

- We further note that $K \equiv h^{ab} K_{ab} = n^\alpha_{;\alpha}$

Physical Interpretation : K is equal to the expansion of the congruence of geodesics that are hypersurf orthogonal. Therefore, $k = \theta$

- h_{ab} is concerned with purely intrinsic aspects of hypersurface.
  k_{ab} is concerned with extrinsic aspects.

Equation of Motion and well Posed Problem

- The trajectory of the system between two instants of time is the extremum of the action integral.
$$A[q_i^1, q_i^2, t_1, t_2] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt$$
- By varying the trajectory, the change in action is
$$\delta \mathcal{A} = \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]$$
- By setting the end points fixed $\delta q = 0$ and making $\delta A = 0$ we get
$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0$$
- Upon expanding the time derivative we get 2nd order DE
$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j$$

Remark : Action Principle tells us what need to be fixed at the boundary without we assuming anything.

Addition of derivative term

- Let $L = L(q, \dot{q}, \ddot{q})$, to get the 2nd order EOM, we have to set $\delta q_i = 0$ and $\delta \dot{q}_i = 0$. etting $\delta q = 0$ at the endpoints is no longer sufficient to kill the boundary term. We are setting 4 boundary conditions for 2nd order EOM. For most choices of boundary data EOM would not render any solution. Addition of total derivative can make the variational principle ill-posed.
- Let Lagrangian be linear in $\ddot{q}$

$$L_1 = L + \tfrac{d}{dt} f(q, \dot{q}, t)$$
$$\delta A_1 = \delta A + \delta f\left(q_2, \dot{q}_2, t_2\right) - \delta f\left(q_1, \dot{q}, t_1\right)$$

- **Example**
$\mathcal{A} = \int_1^2 dt \left(-\frac{1}{2} q\ddot{q}\right) = \int_1^2 dt \left(\frac{1}{2}\dot{q}^2\right) - \int_1^2 dt \frac{d}{dt}\left(\frac{1}{2} q\dot{q}\right)$
- Varying the action we obtain

$$\begin{aligned}\delta\mathcal{A} &= -\int_1^2 dt\delta q\ddot{q} + (\dot{q}\delta q)_1^2 - \frac{1}{2}(\delta q\dot{q} + q\delta\dot{q})_1^2 \\ &= -\int_1^2 dt\delta q\ddot{q} + \frac{1}{2}(\dot{q}\delta q - q\delta\dot{q})_1^2\end{aligned}$$

Remarks on Boundary term

- $\delta\mathcal{A} = -\int_1^2 dt\delta q\ddot{q} + \frac{1}{2}(\dot{q}\delta q - q\delta\dot{q})_1^2$
- Bulk equation is still 2nd order but we have to fix both $\delta\dot{q} = 0$ and $\delta q = 0$ as the boundary conditions which makes the problem ill-posed.
- **Solution**: To make this problem well posed we add boundary terms so that we are left with Lagrangian which does not depend on higher derivatives in the action.
- We will see ahead how this is relevant in the action for gravitational field. This is exactly in analogy with Gravitational action where we add GHY term to make the action well posed.

Introduction

- **Motivation** : Just as the action for scalar/ vector field, action for gravitation can be dependent on dynamical variable and derivative of dynamical variable but no non trivial scalar Lagrangian can be constructed from the metric and its first derivative because in local inertial frame $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $\partial_\gamma g_{\alpha\beta} = 0$. Only choice left with us is to use $\partial_\nu \partial_\mu g_{\alpha\beta}$ in the action but we are further constrained to get 2nd order field equations.
- To get 2nd order differential equation the 2nd derivative of $g_{\alpha\beta}$ should be linear in action. By doing this, we get some boundary term which we can cancel by adding another term to the action, which in this case is GHY boundary term.
- Therefore, the most simple scalar that can be constructed which has second derivative of metric is Ricci scalar which is constructed from the Riemann tensor, which contains second derivatives of the metric.
- Action for gravitational field is

$$16\pi G\mathcal{A} = \int_{\mathcal{V}} d^4x\sqrt{-g}R\left(g, \partial g, \partial^2 g\right)$$

Variation of the action

- Assuming $g_{\alpha\beta}$ the dynamical variable, varying the action w.r.t. it

$$\begin{aligned}\delta A_{\mathrm{EH}} &= \int_{\mathcal{V}} d^4x\sqrt{-g}G_{\alpha\beta}\delta g^{\alpha\beta} + \int_{\mathcal{V}}^4 x\sqrt{-g}\nabla_\alpha \delta n^\alpha \\ &= \int_{\mathcal{V}} d^4x\sqrt{-g}G_{\alpha\beta}\delta g^{\alpha\beta} + \int_{\partial\mathcal{V}} \delta n^\alpha d\Sigma_\alpha\end{aligned}$$

  where $\delta n^\gamma = g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\mu}\delta\Gamma^\alpha_{\alpha\mu}$

- Assuming the boundary is timelike, variation of metric at the boundary vanishes $\delta g_{\alpha\beta} = 0$, due to which variation of the tagential derivative of metric also vanishes $\delta\partial_\mu g_{\alpha\beta} e^{mu}_a = 0$ we get :

$$\begin{aligned}16\pi G\delta\mathcal{A} = &\int_{\mathcal{V}} d^4x\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\delta g^{\alpha\beta} \\ &-\epsilon\int_{\partial\mathcal{V}} d^3y\sqrt{|h|}n^\mu h^{\alpha\beta}\left(\partial_\mu \delta g_{\alpha\beta}\right)\end{aligned}$$

Gauss Theorem: Illustration

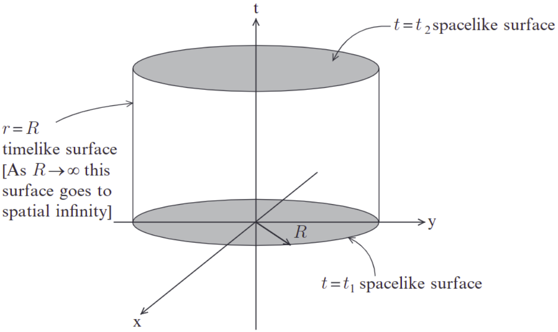

Figure: The Gauss theorem in a spacetime volume is illustrated

Solution - I

- To get the bulk equation we have to fix normal derivatives also but fixing that makes our action principle ill-posed and hence the field equations are not consistent with boundary data.
- Therefore to make gravitational action well posed we have to add a boundary term so that normal derivative part cancels away.

$$A_{GHY} = 2 \int_{\partial \mathcal{M}} \mathrm{d}^3 y \epsilon \sqrt{h} K$$

where K is the trace of extrinsic curvature

> **Remark** : As extrinsic curvature is related to normal derivative of the metric and boundary term is normal derivative of the metric, therefore, trace of extrinsic curvature is not a bad guess.

- Varying the GHY term we can cancel our boundary term and obtain the bulk term.

Well-Posed Gravitational Action

- Now, we will derive the field equations without assuming a priori that variation of the metric vanishes.

  Remark : In earlier calculation we expanded tensor $\delta\Gamma$ to variation of metric terms, and then assumed the variation of the metric vanishes. Here, we will convert $\delta\Gamma$ to covariant derivatives using

$$\delta\left(\nabla_\alpha n_\beta\right) = \nabla_\alpha \delta n_\beta - \left(\delta\Gamma^\gamma_{\alpha\beta}\right) n_\gamma$$
$$\delta\left(\nabla_\alpha n^\alpha\right) = \nabla_\alpha \delta n^\alpha + n^\gamma \delta\Gamma^\alpha_{\alpha\gamma}$$

- By varying Einstein Hilbert Action we get:

$$\delta A_{\text{EH}} = \int_{\mathcal{V}} d^4x\sqrt{-g}\left[G_{\alpha\beta}\delta g^{\alpha\beta}\right] + \int_{\mathcal{V}} d^4x\sqrt{-g}\nabla_\gamma\left[g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\mu}\delta\Gamma^\alpha_{\alpha mu}\right]$$
$$\equiv \int_{\mathcal{V}} d^4x\sqrt{-g}\left[G_{\alpha\beta}\delta g^{\alpha\beta}\right] + \int_{\partial\mathcal{V}} d^3x\sqrt{h}\mathcal{B}\left[n_\gamma\right]$$

where $\mathcal{B}\left[n_\gamma\right] = n_\gamma\left[g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\mu}\delta\Gamma^\alpha_{\alpha\mu}\right]$

$\mathcal{B}\left[n_\gamma\right] = \nabla_\alpha\left(\delta u^\alpha\right) - \delta\left(2\nabla_\alpha n^\alpha\right) + \left(\nabla_\alpha n_\beta\right)\delta g^{\alpha\beta}$

where $\delta u^\alpha \equiv \delta n^\alpha + g^{\alpha\beta}\delta n_\beta$

Well Posed Gravitational Action

- We can easily see δu^{α} lie on the hypersurface. Using the definition of intrinsic covariant derivative, we get $\nabla_a \delta u^a = \nabla_\alpha \delta u^\alpha - \epsilon a_\alpha \delta u^\alpha$ where $a_\alpha = n^\beta \nabla_\beta n_\alpha$; and $\nabla_a \delta u^a$ is the intrinsic covariant derivative.
- Further, by property of $a^\alpha \delta n_\alpha = 0$ we get, $\mathcal{B}\left[n_\gamma\right] = \nabla_a \left(\delta u^a\right) - \delta\left(2\nabla_\alpha n^\alpha\right) + \left(\nabla_\alpha n_\beta - \epsilon n_\alpha a_\beta\right)\delta g^{\alpha\beta}$ where we used the definition of extrinsic curvature $k_{\alpha\beta} = \nabla_\alpha n_\beta - \epsilon n_\alpha n^\gamma \nabla_\gamma n_\beta$
- Finally using the properties of extrinsic curvature we get the boundary term

$$\int_{\partial\mathcal{V}} d^3x\sqrt{h}\mathcal{B}\left[v_c\right] = \int_{\partial\mathcal{V}} d^3x\sqrt{h}\nabla_a\left(\delta u^a\right) + \delta\int_{\partial\mathcal{V}} d^3x 2K\sqrt{h} + \int_{\partial\mathcal{V}} d^3x\sqrt{h}\left(Kh_{\alpha\beta} - K_{\alpha\beta}\right)\delta h^{\alpha\beta}$$

- We have to add $2K\sqrt{h}$ term,which is the GHY term, to cancel the boundary term and get the field equation.
- At the boundary we need not have to fix whole metric but just induced metric h^{ij}

Boundary term in Electrodynamics

- Electrodynamics is the vector field theory where A^i is the 4-vector for which the action for the free Electrodynamic field is given by

$$\mathcal{A} = -\tfrac{1}{16\pi}\int_{\mathcal{V}} F_{ik}F^{ik} d^4x$$

where $\quad F_{ik} = \partial_i A_k - \partial_k A_i$

- By assuming A^i as dynamical variable and varying the action w.r.t it, we get:

$$\begin{aligned}\delta\mathcal{A} &= \delta\left(-\frac{1}{16\pi}\int_{\mathcal{V}} d^4x F_{ab}F^{ab}\right) \\ &= -\frac{1}{4\pi}\int_{\mathcal{V}} d^4x \partial_k F^{ik}\delta A_i - \frac{1}{4\pi}\int_t d^3x \mathbf{E}.\delta\mathbf{A}\end{aligned}$$

- Bulk term will lead to Maxwell equations and to vanish boundary term we have to fix spatial part of vector potential A at the $t = constant$ surfaces. Further, we can show that we have to fix just magnetic field at the $t = constant$ surfaces.

Introduction

- The purpose of 3+1 decompostion is to express the action in terms of the Hamiltonian it is necessary to foliate V with a family of spacelike hypersurface.

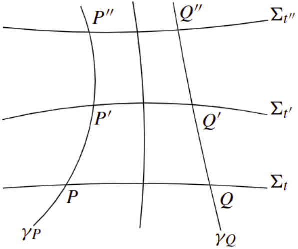

Figure: Foliation of Spacetime by Spacelike Hypersurfaces

- Assuming a scalar field $t(x^{\alpha})$ such that t = constant, and on each hypersurface we assume coordinates y^a. Also assuming y^a is constant along the flow of the curve, which defines the mapping of the point P to P' and so on.

Line Element

- Therefore, the transformation exists $x^\alpha = x^\alpha(t, y^a)$.
- Tangent vector along the curve is $t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a}$

  Tangent vectors on the hypersurface is $e_a^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t$
- Lapse function N, and shift vector N^a can be defined as

$$t^\alpha = Nn^\alpha + N^a e_a^\alpha \text{ and } n_\alpha = -N\partial_\alpha t$$

  where n_α is the normal vector.

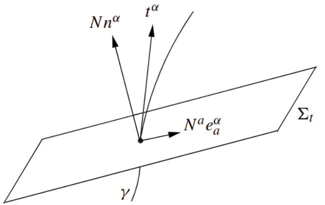

Figure: Decomposition of t^α into lapse and shift vector

Line Element

- The metric in the coordinates (t, y^a) can be expressed as
$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + h_{ab}\left(\,\mathrm{d}y^a + N^a\,\mathrm{d}t\right)\left(\mathrm{d}y^b + N^b\,\mathrm{d}t\right)$$
- Further, we can easily get $\sqrt{-g} = N\sqrt{h}$
- For displacement along the curve, increment in proper time is related to the increment in coordiante time as
$$d\tau^2 = \left(N^2 + h_{ab}N^aN^b\right)dt^2$$
- If the congruence is Hypersurface orthogonal, Shift vectors N^a vanishes and hence proper time would be related to coordinate time by lapse function as:
$$d\tau^2 = \left(N^2\right)dt^2$$

Schematic of Decomposition

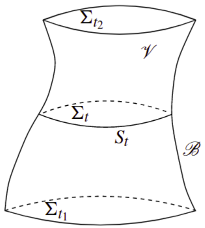

Figure: Decomposition of t^α into lapse and shift vector

- Foliation of V by Σ_t
- S_t embedded in Σ_t
- S_t embedded in V
- B embedded in Spacetime
- Foliation of B by S_t

3+1 Decomposed Action

Surface	Σ_t	S_t	B
Unit normal	n^α	r^α	r^α
Coordinates	y^a	θ^A	z^i
Tangent vectors	e^α_a	e^α_A	e^α_i
Induced metric	h_{ab}	σ_{AB}	γ_{ij}
Extrinsic curvature	K_{ab}	k_{AB}	K_{ij}

- **3+1 Decomposition of the action**

$$S_G = \frac{1}{16\pi}\int_{t_1}^{t_2} \mathrm{d}t \left\{ \int_{\Sigma_t} \left({}^3R + K^{ab}K_{ab} - K^2 \right) N\sqrt{h}\, \mathrm{d}^3y \right.$$
$$\left. + 2\oint_{S_t} (k - k_0)\, N\sqrt{\sigma}\mathrm{d}^2\theta \right\}$$

Remarks

- Let $\dot{h}_{ab} \equiv L_f h_{ab}$, we can show
$$\dot{h}_{ab} = 2NK_{ab} + N_{a|b} + N_{b|a}$$
- Therefore, the action does not involve $\dot{N}$, nor $\dot{N}^a$. The dynamical variable is induced metric only, lapse and shift only serve to specify the foliation of V into spacelike hypersurfaces.
- Because the foliation is arbitrary, we are free to choose lapse and Shifts.
- Momentum conjugate to induced metric is defined as $p^{ab} = \frac{\partial}{\partial \dot{h}_{ab}} (\sqrt{-g} L_G)$ where L_G is the bulk part of lagrangian. It can be shown that
$$(16\pi)p^{ab} = \sqrt{h}\left(K^{ab} - Kh^{ab}\right)$$

Applications

- Singularity Theorems
- Boundary term for Null Hypersurfaces
- Numerical Relativity
- Relativistic Hydrodynamics

References

1. Eric Poisson. A relativist's toolkit: the mathematics of black-hole mechanics. Cambridge university press, 2004.
2. Thanu Padmanabhan. Gravitation: foundations and frontiers. Cambridge University Press, 2010.
3. T Padmanabhan. A short note on the boundary term for the hilbert action. Modern Physics Letters A, 29(08):1450037, 2014.
4. Ethan Dyer and Kurt Hinterbichler. Boundary terms, variational principles, and higher derivative modified gravity. Physical Review D, 79(2):024028, 2009.
5. Krishnamohan Parattu, Sumanta Chakraborty, Bibhas Ranjan Majhi, and T Padmanabhan. A boundary term for the gravitational action with null boundaries. General Relativity and Gravitation, 48(7):1–28, 2016.

Thank You