Geodesic Congruences and Raychaudhuri Equations

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Goal

The goal of the poster is to make the reader familiar with the mathematical tools which are required to understand singularity theorems. Instead of dealing with multitude of examples of singularity theorem, the author feels that intuitive grasp related to these tools would serve the purpose in a better way. Moreover, these mathematical tools forms the basis of the singularity theorems.

Non-Null Geodesics

- Decomposition of $g_{\alpha\beta}$ into longitudinal part $U_\alpha U_\beta$ and transverse part $h_{\alpha\beta}$

$$g_{\alpha\beta} = h_{\alpha\beta} + U_\alpha U_\beta \tag{1}$$

- Properties of transverse metric $h_{\alpha\beta}$
  1. $h_{\alpha\beta}U^\beta = U^\alpha h_{\alpha\beta} = 0$
  2. $h^\alpha_\gamma h^\gamma_\beta = h^\alpha_\beta$
  3. $h^\alpha_\alpha = 3$
- As lie derivative of deviation vector w.r.t vector field vanishes

$$u^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta U^\alpha \tag{2}$$

  where $B_{\alpha\beta} = \nabla_\beta U_\alpha$
- Property of $B_{\alpha\beta}$

$$U^\beta B_{\alpha\beta} = B_{\alpha\beta} U^\alpha = 0 \tag{3}$$

  Hence, $B_{\alpha\beta}$ is purely spatial
- We can show by construction $\xi^\alpha U_\alpha = 0$ showing both are orthogonal to each other
- As $B_{\alpha\beta}$ is spatial we can decompose it as

$$B_{\alpha\beta} = \frac{h_{\alpha\beta}\theta}{3} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{4}$$

  where $\theta = \nabla_\alpha U^\alpha$ is the expansion parameter, $\sigma_{\alpha\beta}$ (spatial) is the shear parameter, and $\omega_{\alpha\beta}$ (spatial) is the rotation parameter
- **Frobenius theorem** : Geodesic is Hypersurface orthogonal iff $\omega_{\alpha\beta} = 0$
- **Raychaudhuri Equation** : Evolution equation for the expansion parameter

$$\frac{d\Theta}{d\lambda} = -\frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 - R_{\alpha\beta}U^\alpha U^\beta \tag{5}$$

- **Focusing Theorem** : In the above equation if the congruence is Hypersurface orthogonal and the matter follows strong energy condition i.e.$R_{\alpha\beta}U^\alpha U^\beta \geq 0$ then the expansion must decrease during the congruence's evolution.
  Physical Interpretation : Gravitation is an attractive force when the strong energy condition holds, and geodesics gets focused as a result of this attraction.

Null Geodesics

- Decomposition of $g_{\alpha\beta}$ into

$$g_{\alpha\beta} = h_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta \tag{6}$$

  where $K_\alpha N^\alpha = 1$
- Properties of transverse metric $h_{\alpha\beta}$
  1. $h_{\alpha\beta}K^\beta = K^\alpha h_{\alpha\beta} = 0$
  2. $h_{\alpha\beta}N^\beta = N^\alpha h_{\alpha\beta} = 0$
  3. $h^\alpha_\gamma h^\gamma_\beta = h^\alpha_\beta$
  4. $h^\alpha_\alpha = 2$
- As lie derivative of deviation vector w.r.t vector field vanishes

$$K^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta K^\alpha \tag{7}$$

  where $B_{\alpha\beta} = \nabla_\beta K_\alpha$
- Properties of $B_{\alpha\beta}$ are
  1. $K^\beta B_{\alpha\beta} = B_{\alpha\beta}K^\alpha = 0$
  2. $B_{\alpha\beta}N^\alpha \neq 0$
  3. $N^\beta B_{\alpha\beta} \neq 0$

  Thereore, $B_{\alpha\beta}$ has non transverse component
- We can show by construction $\xi^\alpha K_\alpha = 0$
  This fails to remove component of ξ^α along K^α and hence, transverse deviation vector can be written as

$$\tilde{\xi^\alpha} = h^\alpha_\beta \xi^\beta \tag{8}$$

  Further by calculation we find the evolution of $\tilde{\xi^\alpha}$ has non-transverse components, thus finding transverse component of it we get

$$\widetilde{K^\beta \nabla_\beta \tilde{\xi^\alpha}} = \tilde{B^\alpha_i}\tilde{\xi^i} \tag{9}$$

- As $\tilde{B_{\alpha\beta}}$ is in transverse space, we can decompose it as

$$\tilde{B_{\alpha\beta}} = \frac{h_{\alpha\beta}\theta}{2} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{10}$$

  where $\theta = \nabla_\alpha K^\alpha$ is the expansion parameter.
- **Frobenius theorem** : Geodesic is Hypersurface orthogonal iff $\omega_{\alpha\beta} = 0$
- **Raychaudhuri Equation**

$$\frac{d\Theta}{d\lambda} = -\frac{1}{2}\Theta^2 - \sigma^2 + \omega^2 - R_{\alpha\beta}K^\alpha K^\beta \tag{11}$$

- **Focusing Theorem** : In the above equation if the congruence is Hypersurface orthogonal and the matter follows strong energy condition (for null congruence Strong energy condition implies weak energy condition) i.e.$R_{\alpha\beta}K^\alpha K^\beta \geq 0$ then the expansion must decrease during the congruence's evolution.

Physical Interpretation

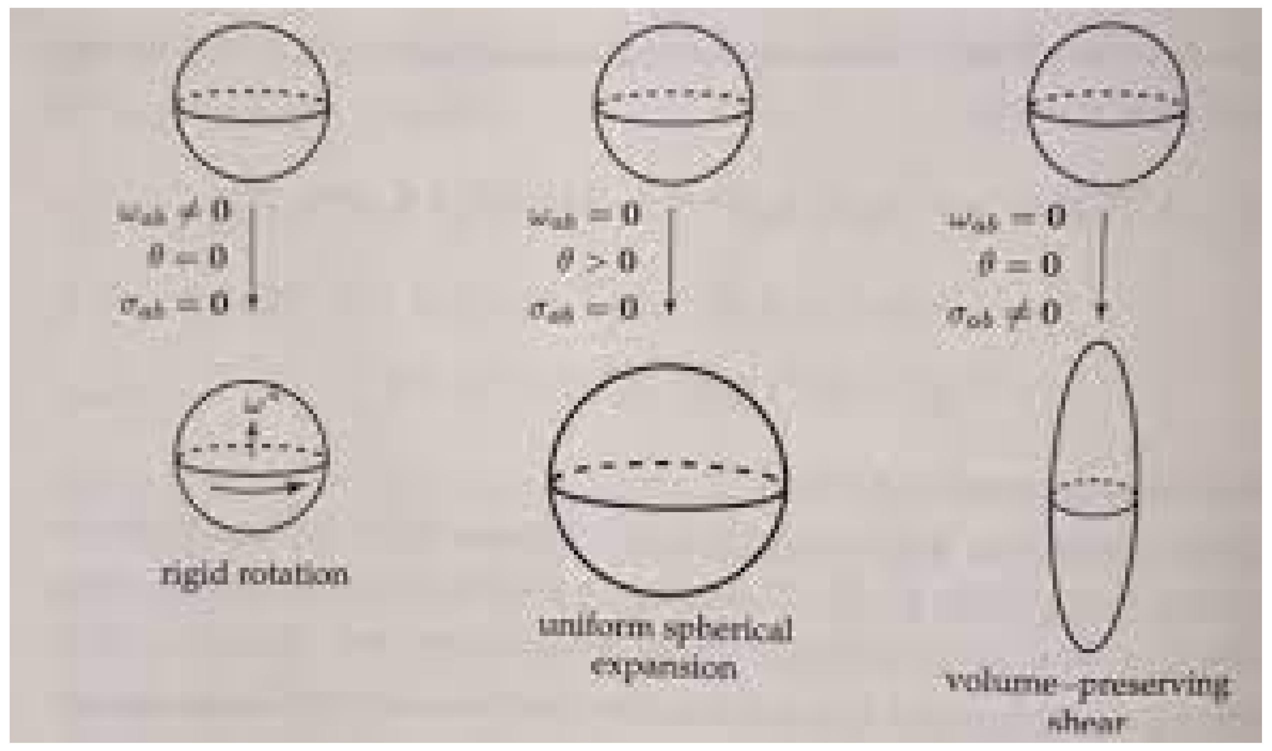

Fig. 1: Graphical representation of Expansion, Shear, Rotation parameter.

Event Horizon

Event horizon is the null hypersurface.There is a property of null hypersurface that the null curves form null geodesics and such congruences are hypersurface orthogonal. On these null hypersurfaces, to ease the calculations we take the coordinates as y_i, where

$$e^\alpha_i = \frac{\partial x^\alpha}{\partial y^i} = (k^\alpha, \theta_A, \theta_B)$$

By construction we can impose:

$$k_\alpha e^\alpha_A = 0; k_\alpha e^\alpha_B = 0 \tag{12}$$

As $e^\alpha_1 = K^\alpha$, therefore we have 3 basis vectors. To cover the whole spacetime we have to introduce one more basis vector N^α with 4 conditions:
1. $N^\alpha N_\alpha = 0$
2. $N^\alpha K_\alpha = 1$
3. $N^\alpha e^\alpha_A = 0$

which yields unique N^α

Remarks

The main distinguishable point between null hypersurfaces and non-null hypersurfaces is that in the former, the transverse metric is 2 dimensional, and in the latter, the transverse metric is 3 dimensional.

References

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- Padmanabhan, T. (2010). Gravitation: Foundations and Frontiers. Cambridge: Cambridge University Press. doi:10.1017/CBO9780511807787